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Executive Summary

This report describes the research activity in the Seventh Framework Programme, Theme 3 "Information and Communication Technologies", STREP research project **Hierarchical and Distributed Model Predictive Control of Large Scale Systems- HD-MPC**, focusing on WP5 - "Distributed state estimation algorithms". Specifically, the report aims at presenting the main results achieved in Task 5.1 (State estimation) and Task 5.2 (Variance estimation).
The report is organized in four chapters:

- Chapter 1 presents a synopsis of the report, summarizes the content of the following chapters and, for each one of them, highlights the main results achieved.
- Chapter 2 generalizes to nonlinear discrete-time systems, the results, already described in Deliverable D5.2, concerning the problem of distributed state estimation, i.e. the problem of estimating the state of the system by means of a network of sensors that can exchange information according to a given topology. Then, the problem is formally stated and a solution based on the use of moving horizon estimators (MHE) is proposed.
- Chapter 3 presents an MHE method for discrete-time nonlinear partitioned systems, i.e. systems decomposed into coupled subsystems with non-overlapping states. In the proposed algorithm, each subsystem solves a reduced-order MHE problem to estimate its own state based on the estimates computed by its neighbors. Conditions for the convergence of the estimates are investigated. The algorithm is applied to the model of three river reaches, i.e. to a part of the hydropower valley extensively studied within the HD-MPC project.
- Chapter 4 deals with the problem of variance estimation. This problem is of paramount importance since the distributed estimation approaches described in the previous deliverables and chapters require the a-priori knowledge of the covariances of the noises affecting the system states and outputs, which are generally unknown. Therefore, a simple on-line covariance estimation algorithm is developed and its performance analyzed in a couple of significant test cases.

Chapter 1

Synopsis of the report

1.1 Synopsis of Chapter 2

First of all, it is worth recalling that in deliverable D5.1 a distinction was made between *distributed estimation*, where each agent estimates the state of the whole system, and *partition-based estimation*, where each agent estimates only part of the whole state based on its own measurements and on the information transmitted by its neighborhoods, including the estimates of other system's components. In this final deliverable, this distinction will be reconsidered and two specific algorithms, one referred to the *distributed estimation* problem (Chapter 2) and the other referred to the *partition-based estimation* problem (Chapter 3), will be described. Both of them are based on the moving horizon estimation (MHE) approach and are derived for nonlinear discrete-time systems.
Specifically, in this chapter, the previous results on distributed state estimation for linear systems already described in Deliverable D5.2, are generalized to nonlinear systems. The goal is to provide a Nonlinear DMHE (NDMHE) scheme enjoying stability properties. In order to characterize states that can and cannot be recovered by each sensor without communication the notion of MHE detectability, see [43], is first exploited. Moreover, a consensus-on-estimates penalty term in local MHE problems is used to let each sensor learn locally MHE undetectable parts of the state from other sensors. The state estimation error dynamics is derived and it is shown that when it enjoys incremental input-to-state stability (δISS) [6], so that stability of the estimation scheme is guaranteed. Unfortunately, checking δISS properties can be hard and requires a global analysis of all estimation errors committed by individual sensors. Therefore, exploiting a small gain property inspired by [10], simple conditions are provided on the weights associated to communication channels in order to enforce stability of DMHE. An example of application concerning four Van der Pol oscillators is considered to analyze the performance of the proposed NDMHE algorithm. All the technical proofs of the main results are collected at the end of the chapter.

1.2 Synopsis of Chapter 3

The distributed approach described in the previous chapter is mainly useful for the coordination of sensor networks, i.e. when each sensor must estimate the whole system state with the limited information provided by its own measurements and those transmitted by its neighboring sensors. However, in industrial process control, the distributed estimation problem can assume significantly different characteristics. In fact, many industrial processes and physical systems are composed by a large number of interconnected units, each one described by a dynamic model. In these cases, the computational load associated to the design of a unique centralized controller can be high; moreover, a centralized approach does not take advantage of the sparsity of the system. For these reasons, within the HD-MPC project research has focused on the design of efficient and reliable distributed control systems (see WP3 and WP4), which, however, are usually state-feedback. Therefore, in order to guarantee a fully distributed control design, also distributed state estimation algorithms dealing with constraints are needed.

Early works in this field have been reported [30], where a solution based on the use of reduced-order and decoupled models for each subsystem was proposed, while subsystems with overlapping states were considered in [25, 50, 49, 51]. Within the HD-MPC project, three partition-based MHE algorithms (PMHE) for linear constrained systems decomposed into interconnected subsystems without overlapping states have recently been developed and described in [17]. In these algorithms, which differ in terms of communication requirements, accuracy and computational complexity, each subsystem solves a reduced-order MHE problem in order to estimate its own states based on the estimate of the other subsystems' states transmitted by its neighbors.

In this chapter, the results of [17] are extended to the case of nonlinear systems so as to cope with the majority of problems arising in process control, where the nonlinear dynamic phenomena have often to be considered in order to guarantee the accuracy of the solution. The convergence properties of the method are investigated and sufficient conditions are given. These conditions turn out to be automatically satisfied when the directed graph describing interconnections among subsystems is acyclic.

The proposed partition-based MHE is applied to the problem of estimating the levels and flow rates in the model of three cascade river reaches, which represents a part of the Hydro Power Valley benchmark extensively studied in the HD-MPC project (WP7). Interconnections between successive reaches are due to the dependence of the input flow rate of the downstream reaches to the level of the final section of the upstream ones, which cannot be measured, but just estimated from the available measures collected along the reach.

All the technical proofs of the main results are collected at the end of the chapter.

1.3 Synopsis of Chapter 4

All the distributed estimation algorithms base on the MHE approach developed in the HD-MPC project and partially described in the previous chapters of this deliverable require the a-priori knowledge of the covariances of the noises affecting the system states and outputs, which are generally unknown. This is a serious drawback which could prevent one from achieving satisfactory results, and a particular attention must be placed to the tuning phase.

Many different approaches have been proposed in the technical literature to solve the problem of covariance estimation; some of them have already been presented and analyzed in deliverable D5.2. The analysis reported there has shown that the so-called correlation approach is probably the most effective and reliable one. Therefore, the algorithms developed by Mehra, see [28, 29], and the Autocovariance Least Squares (ALS) method described in [31] have been specifically considered. Further tests have proven that the ALS approach is the most effective one, since it outperforms significantly the one proposed in [28].

In this chapter the ALS method is used to implement a novel adaptive algorithm for the on-line estimation of the noise properties; the estimates so computed are then used for the adaptive tuning of the weights of the moving horizon estimators. Basically, starting form the output estimation error computed on-line, this algorithm adaptively updates the noise variances, which correspond to the inverse of the weights in the MHE performance index. The method is used in a couple of significant test cases with excellent results, so that it is believed that it can be successfully applied in the majority of cases. As a simple example, extensively discussed in the chapter, consider a system affected by a noise with covariance $Q_w = qI$ acting on the state and by a measurement noise with covariance $R_v = rI$; Figure 1.1 shows the estimate of the parameters q and r provided by the method here developed. It is apparent that the unknown covariances are properly estimated, and can be uses in the state estimation algorithms (either Kalman estimators or MHE estimators) to improve their performance.

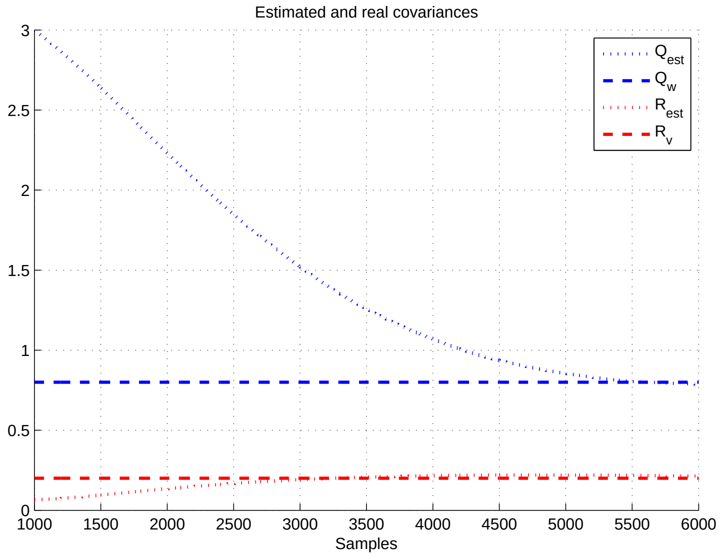

Figure 1.1: Convergence of the covariances via the adaptive law.

The chapter is organized as follows. First, the problem is stated, the performance index used in MHE is recalled and some preliminary definitions are reported. Then the ALS algorithm, already extensively presented in deliverable D5.2, is briefly summarized. The adaptive method for the on-line update of the MHE tuning parameters is then given and tested in two significant simulation examples.

Chapter 2

Distributed moving horizon estimation for nonlinear systems

In this chapter, the problem of distributed state estimation for nonlinear discrete-time systems is formally posed and a solution is proposed with the MHE approach. The content of this chapter is based on the paper [18].

2.1 Introduction

State estimation for nonlinear systems based on distributed sensing schemes is a challenging problem, the solution of which is of great importance in many fields. Distributed monitoring, exploration, surveillance and tracking of moving objects over specific regions are topical applications, due to the wide diffusion of sensor networks in the last decade. Sensor networks are collections of small, low power consuming and possibly cheap sensing devices, with communication and computation capabilities.

Available methods for distributed state estimation rely on local state estimators for linear systems combined with consensus and sensor-fusion algorithms. Typically, each sensor provides an estimate of the system state based on local data and consensus schemes are employed either to provide a wider set of measurement data for each individual sensor (*i.e.*, *consensus on measurements*) or to correct individual state estimates by comparison with neighboring nodes information (*i.e.*, *consensus on estimates*). Approaches to distributed estimation based on Kalman filters have been proposed in [8], [5], [36], [33], [47], [34], [24], [35]. The algorithms described in [36], [33] and [47] rely on consensus on measurements, while in [34] a solution based on consensus on estimates is proposed. Recently, convergence in mean of the local state estimates obtained with the algorithm presented in [33] has been proved in [24] provided that the observed process is stable. Moreover, a stability analysis of the state estimator presented in [34] is provided in [35]. A two-step optimization procedure relying on consensus on estimates is used in [5] and in [8]. In the latter, formal analysis of the estimator properties is carried out when the observed signal is a random walk.
As shown in [42] for centralized estimation problems, methods based on Kalman filtering may become suboptimal or even unstable when constraints on noise and state variables are present. This motivated the development of centralized MHE schemes for linear [40, 2], nonlinear [41, 43, 3, 4] and hybrid [19] systems, capable to guarantee observer convergence and/or stability in a constrained setting. For these reasons, distributed MHE (DMHE) methods for linear constrained systems have been developed in the HD-MPC project; the main results achieved have been described in deliverable D5.2 and in [13]

and [14].

In the following, we generalize our previous results to the nonlinear setting with the goal of providing a Nonlinear DMHE (NDMHE) scheme enjoying stability properties. In order to characterize states that can and cannot be recovered by each sensor without communication we exploit the notion of MHE detectability [43]. Moreover we use a consensus-on-estimates penalty term in local MHE problems to let each sensor learn locally MHE undetectable parts of the state from other sensors. Furthermore, we explicitly derive the error dynamics and show that when it enjoys incremental input-to-state stability (δISS) [6], stability of the estimation scheme is guaranteed. Unfortunately, checking δISS properties can be hard and requires a global analysis of all estimation errors committed by individual sensors. Therefore, exploiting a small gain property inspired by [10], we provide simple conditions on weights associated to communication channels for enforcing stability of DMHE.

The chapter is structured as follows. In Section 2.2 we introduce the observed dynamical system, the structure of the sensor network, and we recall notions of detectability for nonlinear systems. In Section 2.3 we describe the distributed state estimation algorithm. In Section 2.4 we investigate the stability and convergence properties of the presented observer and in Section 2.5 we generalize the results provided in the previous sections to a wider class of systems. In Section 2.6 we show an example of application of the proposed NDMHE algorithm. For the sake of clarity, the proofs are collected at the end of the chapter.

Notation. In the chapter, the following notation will be used. I_n and $\mathbf{0}_{\nu\times\mu}$ denote the $n\times n$ identity matrix and the $\nu\times\mu$ matrix of zero elements, respectively. The symbol $\otimes$ denotes the Kronecker product, and $\mathbb{1}_M$ is the M-dimensional column vector whose entries are all equal to 1. The matrix $\mathrm{diag}(M_1,\dots,M_s)$ is block-diagonal with blocks M_i. We use the short-hand $\mathbf{v}=(v_1,\dots,v_s)$ to denote a column vector with s (not necessarily scalar) components. For a discrete-time signal $w(t)$ and $a,b\in\mathbb{N}$, $a\le b$, we denote $(w(a),w(a+1),\dots,w(b))$ with $w_{[a:b]}$. For the definition of positive-definite, $\mathscr{K}$, $\mathscr{K}_\infty$ and $\mathscr{KL}$ functions we defer the reader to [43]. Finally, the notation $\|z\|_S^2$ stands for z^TSz, where S is a symmetric positive-semidefinite matrix.

2.2 The system and its observability properties

2.2.1 System and sensor network

We assume that the observed process obeys to the dynamics

$$x_{t+1}=f(x_t,w_t) \tag{2.1}$$

where $x_t\in\mathbb{X}\subseteq\mathbb{R}^n$ is the state vector and the term $w_t\in\mathbb{W}\subseteq\mathbb{R}^m$ represents an unknown disturbance term. We assume that the sets $\mathbb{X}$ and $\mathbb{W}$ are convex and that $\mathbb{W}$ contains the origin. Furthermore, $f(x,w)$ has continuous partial derivatives with respect to the components w^j of w, $j=1,\dots,n$, and satisfies the following Assumption.

Assumption 1 *Function f is globally Lipschitz with respect to w and with respect to x* i.e., $\exists l>0:$ $\forall x\in\mathbb{X}$ *and* $\forall w_1,w_2\in\mathbb{W}$

$$\|f(x,w_1)-f(x,w_2)\|\le l\|w_1-w_2\| \tag{2.2a}$$

and $\exists l_x>0\ \forall x_1,x_2\in\mathbb{X}$

$$\|f(x_1,0)-f(x_2,0)\|\le l_x\|x_1-x_2\| \tag{2.2b}$$

Measurements on the state vector are performed by M sensors, according to the sensing models (in general different from sensor to sensor)

$$y_t^i = h^i(x_t) + v_t^i \ , \ i = 1, ..., M \tag{2.3}$$

where the term $v_t^i \in \mathbb{R}^{p_i}$ represents an unknown measurement error.

The communication network among sensors is modeled by the directed graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, where the nodes in $\mathscr{V} = \{1, 2, \dots, M\}$ are sensors and an edge (j, i) in the set $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$ models that sensor j can transmit information to sensor i. We assume $(i, i) \in \mathscr{E}, \forall i \in \mathscr{V}$. We denote with $\mathscr{V}_i$ the set of the neighbors to node i, *i.e.*, $\mathscr{V}_i = \{j \in \mathscr{V} : (j, i) \in \mathscr{E}\}$.

We associate to the graph $\mathscr{G}$ the stochastic matrix $K \in \mathbb{R}^{M \times M}$, with entries

$$k_{ij} \geq 0 \text{ if } (j, i) \in \mathscr{E} \tag{2.4a}$$

$$k_{ij} = 0 \text{ otherwise} \tag{2.4b}$$

$$\sum_{j=1}^{M} k_{ij} = 1, \ \forall i = 1, ..., M \tag{2.4c}$$

Any matrix K with entries satisfying (2.4) is said to be *compatible* with $\mathscr{G}$. At a generic time instant t, sensor i collects measurements produced by itself and its neighboring sensors. Moreover, each sensor transmits and receives information once within a sampling interval *i.e.*, measurements available to node i are y_t^j, with $j \in \mathscr{V}_i$.

Three types of quantities can be distinguished: *individual*, *regional*, and *collective*. Specifically, a quantity is referred to as: (*a*) *individual* (with respect to sensor i) when it is related to the node i solely; (*b*) *regional* (with respect to sensor i) if it is related to the nodes in $\mathscr{V}_i$; (*c*) *collective*, if it is related to the whole network. For the sake of clarity, we use different notations for individual, regional and collective variables. Namely, given a variable z, z^i, $\overline{z}^i$ and $\mathbf{z}$ represent its individual, regional and collective version, respectively. For instance, we refer to y_t^i in (2.3) as individual measurement. On the other hand, if $\mathscr{V}_i = \{j_1^i, ..., j_{\nu_i}^i\}$, the regional measurement of node i is given by

$$\overline{y}_t^i = \overline{h}^i(x_t) + \overline{v}_t^i \tag{2.5}$$

where $\overline{y}_t^i = (y_t^{j_1^i}, \dots, y_t^{j_{\nu_i}^i})$, $\overline{h}^i(x_t) = (h^{j_1^i}(x_t), \dots, h^{j_{\nu_i}^i}(x_t))$, and $\overline{v}_t^i = (v_t^{j_1^i}, \dots, v_t^{j_{\nu_i}^i})$. The dimension of vectors $\overline{y}_t^i$ and $\overline{v}_t^i$ is $\overline{p}_i = \sum_{k=1}^{\nu_i} p_{j_k^i}$.

2.2.2 Detectability properties

For the sake of simplicity, we assume that the state vector of systems (2.1), (2.5) can be split into two sub-vectors $x_t^{D,i} \in \mathbb{R}^{n_D^i}$ and $x_t^{UD,i} \in \mathbb{R}^{n - n_D^i}$, whose dynamics is given by

$$x_{t+1}^{UD,i} = f^{UD,i}(x_t^{UD,i}, x_t^{D,i}, w_t^i) \tag{2.6a}$$

$$x_{t+1}^{D,i} = f^{D,i}(x_t^{D,i}, w_t^i) \tag{2.6b}$$

$$\overline{y}_t^i = \tilde{h}^i(x_t^{D,i}) + \overline{v}_t^i \tag{2.6c}$$

where the subsystem (2.6b), (2.6c) is MHE detectable, according to the notion of MHE detectability for nonlinear systems introduced in [43]. In Section 2.5 we will show how to generalize the main results to systems (2.1)-(2.5) that can be brought in the form (2.6) via a coordinate change.

Definition 1 *The system* $x_{k+1}^{D,i} = f^{D,i}(x_k^{D,i}, w_k^1), y_k = \tilde{h}(x_k^{D,i})$ *is* MHE detectable *if the system augmented with an extra disturbance* w_k^2

$$x_{k+1}^{D,i} = f^{D,i}(x_k^{D,i}, w_k^1) + w_k^2 \tag{2.7a}$$

$$y_k = \tilde{h}^i(x_k^{D,i}) \tag{2.7b}$$

is incrementally input-output-to-state-stable (δIOSS) with respect to the augmented disturbances $\tilde{w}_k = (w_k^1, w_k^2)$*. Namely, there exist functions* $\beta_D \in \mathscr{KL}$*,* $\gamma_1, \gamma_2 \in \mathscr{K}$ *such that, for every two initial states* z *and* z^* *and two disturbance sequences* $\tilde{w}_{[0:k]}$ *and* $\tilde{w}^*_{[0:k]}$ *and, given the corresponding output sequences* $y_{[0:k]}$ *and* $y^*_{[0:k]}$*, it holds that*

$$\|x_t^{D,i} - x_t^{D,i*}\| \leq \beta_D(\|z - z^*\|, t) + \gamma_1(\|\tilde{w}_k - \tilde{w}_k^*\|_{[0:t-1]}) + \\ + \gamma_2(\|y_k - y_k^*\|_{[0:t]})$$

where $x_k^{D,i}$ *and* $x_k^{D,i*}$ *are the state sequences stemming (through system* (2.7a)*) from* z*,* $\tilde{w}_{[0:k]}$ *and from* z^**,* $\tilde{w}^*_{[0:k]}$*, respectively.* □

It follows that $x_t^{D,i}$ and $x_t^{UD,i}$ denote regionally MHE detectable and regionally MHE undetectable components of x_t, respectively. Let $\bar{P}_{UD}^i$ and $\bar{P}_D^i$ be the $n \times n$ orthonormal projection matrices defined in such a way that $\bar{P}_{UD}^i x_t = (x_t^{UD,i}, 0)$ and $\bar{P}_D^i x_t = (0, x_t^{D,i})$, respectively. Note that vectors x_t and $(x_t^{UD,i}, x_t^{D,i})$ are the same up to a permutation of the elements of x_t. Therefore, we define the permutation matrix $\tilde{P}^i = (\bar{P}_{UD}^i + \bar{P}_D^i)^T$ that gives $x_t = \tilde{P}^i(x_t^{UD,i}, x_t^{D,i})$. We define also $\mathbf{P}_{UD} = \text{diag}(\bar{P}_{UD}^1, \ldots, \bar{P}_{UD}^M)$, $\mathbf{P}_D = \text{diag}(\bar{P}_D^1, \ldots, \bar{P}_D^M)$ and $\tilde{\mathbf{P}} = \text{diag}(\tilde{P}^1, \ldots, \tilde{P}^M)$.

Four different notions of MHE detectability can be introduced.

Definition 2 *The system is* individually MHE detectable *by sensor i (sensor i is individually MHE detectable) if the system* $x_{k+1} = f(x_k, w_k)$*,* $y_k^i = h^i(x_k)$ *is MHE detectable. The system is* regionally MHE detectable *by sensor i (sensor i is regionally MHE detectable) if the system* $x_{k+1} = f(x_k, w_k)$*,* $\bar{y}_k^i = \bar{h}^i(x_k)$ *is MHE detectable. The system is* MHE detectable *by a subgraph* $\mathscr{G}^* = (\mathscr{V}^*, \mathscr{E}^*)$*, where* $\mathscr{V}^* = \{i_1, \ldots, i_{M^*}\}$ *(the subgraph* $\mathscr{G}^*$ *is MHE detectable) if the system* $x_{k+1} = f(x_k, w_k)$*,* $\mathbf{y}_k^* = \mathbf{h}^*(x_k)$*, where* $\mathbf{y}_k^* = (\bar{y}_k^{i_1}, \ldots, \bar{y}_k^{i_{M^*}})$ *and* $\mathbf{h}^*(x_k) = (\bar{h}^{i_1}(x_k), \ldots, \bar{h}^{i_{M^*}}(x_k))$ *is MHE detectable. Finally the system is collectively MHE detectable if the graph* $\mathscr{G}$ *is MHE detectable.* □

Note that, in view of Assumption 1, for all $i \in \mathscr{V}$, function $f^{UD,i}$ is globally Lipschitz with respect to $x^{UD,i}$ *i.e.*, $\exists l_x^i > 0\ \forall x_1, x_2 \in \mathbb{X}$ such that $\bar{P}_D^i x_1 = \bar{P}_D^i x_2 = x^{D,j}$ and denoting $x_1^{UD,i} = \bar{P}_{UD}^i x_1$, $x_2^{UD,i} = \bar{P}_{UD}^i x_2$

$$\|f^{UD,i}(x_1^{UD,i}, x^{D,i}, 0) - f^{UD,i}(x_2^{UD,i}, x^{D,i}, 0)\| \leq l_x^i \|x_1^{UD,i} - x_2^{UD,i}\| \tag{2.8}$$

In the sequel we denote as $\mathscr{V}_D$ (respectively as $\mathscr{V}_{UD}$) the set of regionally MHE detectable (respectively regionally MHE undetectable) sensors.

Notice that, for a given sensor i, individual detectability implies regional detectability, and regional detectability of any sensor implies collective detectability, while all opposite implications are false. Furthermore, if the system is regionally MHE detectable by sensor i, then $\bar{P}_{UD}^i = \mathbf{0}_{n \times n}$, $\bar{P}_D^i = I_n$ and $l_x^i = 0$.

2.3 The distributed estimation algorithm

Our aim is to design, for a generic sensor $i \in \mathscr{V}$, an algorithm for computing an estimate of the system state based on regional measurements $\bar{y}_t^i$ and further pieces of information provided by sensors $j \in \mathscr{V}_i$.

The proposed solution relies on the use of MHE, [40, 41, 38, 4, 20, 39], in view of its capability to handle state and noise constraints. More specifically, we propose a Distributed MHE scheme for nonlinear systems (NDMHE) where each sensor solves an individual MHE problem.

2.3.1 The individual minimization problem

Each node $i \in \mathscr{V}$, for a given estimation horizon $N \geq 1$, at time t determines the estimates $\hat{x}^i$ and $\hat{w}^i$ of x and w, respectively, by solving the constrained minimization problem *MHE-i* defined as

$$\Theta_t^{*i} = \min_{\hat{x}^i_{t-N}, \{\hat{w}^i_k\}^t_{k=t-N}} J^i(t-N, t, \hat{x}^i_{t-N}, \hat{w}^i, \hat{\bar{v}}^i, \Gamma^i_{t-N}) \tag{2.9}$$

under the constraints

$$\hat{x}^i_{k+1} = f(\hat{x}^i_k, \hat{w}^i_k), \; k = t-N, \dots, t \tag{2.10a}$$
$$\bar{y}^i_k = \bar{h}^i(\hat{x}^i_k) + \hat{\bar{v}}^i_k \tag{2.10b}$$
$$\hat{w}^i_k \in \mathbb{W} \tag{2.10c}$$
$$\hat{x}^i_k \in \mathbb{X} \tag{2.10d}$$

The individual cost function J^i is given by

$$J^i(t-N, t, \hat{x}^i_{t-N}, \hat{w}^i, \hat{\bar{v}}^i, \Gamma^i_{t-N}) = \sum_{k=t-N}^{t} L^i(\hat{\bar{v}}^i_k, \hat{w}^i_k) + \Gamma^i_{t-N}(\hat{x}^i_{t-N}) \tag{2.11a}$$
$$\Gamma^i_{t-N}(\hat{x}^i_{t-N}) = \Gamma^{C,i}_{t-N}(\hat{x}^i_{t-N}; \hat{\bar{x}}^i_{t-N/t-1}) + \Gamma^{0,i}_{t-N}(\hat{x}^i_{t-N}; \hat{x}^i_{t-N/t-1}) + \Theta^{*i}_{t-1} \tag{2.11b}$$

We denote with $\hat{x}^i_{t-N/t}$ and with $\left\{\hat{w}^i_{k/t}\right\}^t_{k=t-N}$ the optimizers to (2.9) and with $\hat{x}^i_{k/t}$, $k = t-N, ..., t$ the individual state sequence stemming from $\hat{x}^i_{t-N/t}$ and $\left\{\hat{w}^i_{k/t}\right\}^t_{k=t-N}$. Furthermore

$$\hat{\bar{x}}^i_{k/t} = \sum_{j=1}^{M} k_{ij} \hat{x}^j_{k/t} \tag{2.12}$$

denotes the weighted average of the state estimates produced by sensors $j \in \mathscr{V}^i$. In (2.11), the function L^i is the stage cost, $\Gamma^{C,i}_{t-N}$ is the consensus initial penalty and $\Gamma^{0,i}_{t-N}$ is the regularization initial cost. They should be defined in order to satisfy the following assumption.

Assumption 2 *The stage costs L^i and the initial penalties $\Gamma^{C,i}_{t-N}$ and $\Gamma^{0,i}_{t-N}$ are continuous, bounded, positive definite and they satisfy the following inequalities for all $w \in \mathbb{R}^m$, $\bar{v} \in \mathbb{R}^{\bar{p}_i}$, $\hat{x}^i_0, \hat{\bar{x}}^i_{0/N-1} \in \mathbb{R}^n$*

$$\underline{\gamma}_L(\|(\bar{v}, w)\|) \leq L^i(\bar{v}, w) \tag{2.13a}$$
$$\Gamma^{C,i}_0(\hat{x}^i_0; \hat{\bar{x}}^i_{0/N-1}) \leq \gamma_0(\|\hat{x}^i_0 - \hat{\bar{x}}^i_{0/N-1}\|) \tag{2.13b}$$
$$\Gamma^{0,i}_0(\hat{x}^i_0; \hat{x}^i_{0/N-1}) \leq \gamma_0(\|\hat{x}^i_0 - \hat{x}^i_{0/N-1}\|) \tag{2.13c}$$

where $\underline{\gamma}_L$ and γ_0 are suitable $\mathscr{K}_\infty$ functions.

As recalled in [41] and [13], choices of $\Gamma^{C,i}_{t-N}$ and $\Gamma^{0,i}_{t-N}$ fulfilling Assumption 2 are the quadratic functions $\Gamma^{C,i}_{t-N} = \|\hat{x}^i_{t-N} - \hat{\bar{x}}^i_{t-N/t-1}\|^2_{(\Pi^{C,i}_{t-N})^{-1}}$ and $\Gamma^{0,i}_{t-N} = \|\hat{x}^i_{t-N} - \hat{x}^i_{t-N/t-1}\|^2_{(\Pi^{0,i}_{t-N})^{-1}}$ where the matrices $\Pi^{C,i}_{t-N}$ and $\Pi^{0,i}_{t-N}$ must be positive definite and bounded.

The penalty term $\Gamma^{C,i}_{t-N}$ embodies a *consensus-on-estimates* term, in the sense that it penalizes deviations of $\hat{x}^i_{t-N}$ from $\hat{\bar{x}}^i_{t-N/t-1}$. Consensus, besides increasing accuracy of the individual estimates, is fundamental to guarantee convergence of the state estimates to the state of the observed system even if regional MHE detectability does not hold. In other words, it allows sensor i to reconstruct components of the state that cannot be estimated by the i-th regional model.

Finally notice that, since the cost (2.11) and the constraints (2.10) depend only upon regional variables, the overall estimation scheme is decentralized.

2.3.2 The collective minimization problem

The individual estimation problem (2.9) can be given a collective form. To this end, let $\mathbf{J}$ be the collective cost function given by

$$\mathbf{J} = \sum_{i=1}^{M} J^i(t-N, t, \hat{x}^i_{t-N}, \hat{w}^i, \hat{\bar{v}}^i, \Gamma^i_{t-N}) \tag{2.14}$$

Define the collective vectors $\hat{\mathbf{x}}_t = (\hat{x}^1_t, \ldots, \hat{x}^M_t)$, $\hat{\bar{\mathbf{v}}}_t = (\hat{\bar{v}}^1_t, \ldots, \hat{\bar{v}}^M_t)$, $\hat{\mathbf{w}}_t = (\hat{w}^1_t, \ldots, \hat{w}^M_t)$, the matrix $\mathbf{K} = K \otimes I_n$, the quantity $\mathbf{\Theta}^*_t = \sum_{i=1}^{M} \Theta^{*i}_t$ and the collective costs

$$\mathbf{L}(\hat{\bar{\mathbf{v}}}_k, \hat{\mathbf{w}}_k) = \sum_{i=1}^{M} L^i(\hat{\bar{v}}^i_k, \hat{w}^i_k) \tag{2.15a}$$

$$\mathbf{\Gamma}^C_{t-N}(\hat{\mathbf{x}}_{t-N}; \mathbf{K}\hat{\mathbf{x}}_{t-N/t-1}) = \sum_{i=1}^{M} \Gamma^{C,i}_{t-N}(\hat{x}^i_{t-N}; \hat{\bar{x}}^i_{t-N/t-1}) \tag{2.15b}$$

$$\mathbf{\Gamma}^0_{t-N}(\hat{\mathbf{x}}_{t-N}; \hat{\mathbf{x}}_{t-N/t-1}) = \sum_{i=1}^{M} \Gamma^{0,i}_{t-N}(\hat{x}^i_{t-N}; \hat{x}^i_{t-N/t-1}) \tag{2.15c}$$

$$\mathbf{\Gamma}_{t-N}(\hat{\mathbf{x}}_{t-N}) = \mathbf{\Gamma}^C_{t-N}(\hat{\mathbf{x}}_{t-N}; \mathbf{K}\hat{\mathbf{x}}_{t-N/t-1}) + \mathbf{\Gamma}^0_{t-N}(\hat{\mathbf{x}}_{t-N}; \hat{\mathbf{x}}_{t-N/t-1}) + \mathbf{\Theta}^*_{t-1} \tag{2.15d}$$

where $\mathbf{K} = K \otimes I_n$. Then, the collective cost function $\mathbf{J}$ can be rewritten as

$$\mathbf{J}(t-N, t, \hat{\mathbf{x}}_{t-N}, \hat{\mathbf{w}}, \hat{\bar{\mathbf{v}}}, \mathbf{\Gamma}_{t-N}) = \sum_{k=t-N}^{t} \mathbf{L}(\hat{\bar{\mathbf{v}}}_k, \hat{\mathbf{w}}_k) + \mathbf{\Gamma}_{t-N}(\hat{\mathbf{x}}_{t-N}) \tag{2.16}$$

Defining $\mathbf{f}(\hat{\mathbf{x}}_k, \hat{\mathbf{w}}_k) = (f(\hat{x}^1_k, \hat{w}^1_k), \ldots, f(\hat{x}^M_k, \hat{w}^M_k))$, $\bar{\mathbf{y}}_k = (\bar{y}^1_k, \ldots, \bar{y}^M_k)$ and $\mathbf{h}(\hat{\mathbf{x}}_k) = (\bar{h}^1(\hat{x}^1_k), \ldots, \bar{h}^M(\hat{x}^M_k))$, also the constraints (2.10) can be written in the following collective form

$$\hat{\mathbf{x}}_{k+1} = \mathbf{f}(\hat{\mathbf{x}}_k, \hat{\mathbf{w}}_k), \; k = t-N, \ldots, t \tag{2.17a}$$

$$\bar{\mathbf{y}}_k = \mathbf{h}(\hat{\mathbf{x}}_k) + \hat{\bar{\mathbf{v}}}^i_k \tag{2.17b}$$

$$\hat{\mathbf{w}}_k \in \mathbb{W}^M \tag{2.17c}$$

$$\hat{\mathbf{x}}_k \in \mathbb{X}^M \tag{2.17d}$$

It is easy to show that solving the problem

$$\Theta^*_t = \min_{\hat{\mathbf{x}}_{t-N}, \{\hat{\mathbf{w}}_k\}^t_{k=t-N}} \left\{ \mathbf{J}(t-N, t, \hat{\mathbf{x}}_{t-N}, \hat{\mathbf{w}}, \hat{\bar{\mathbf{v}}}, \mathbf{\Gamma}_{t-N}) \text{ subj. to (2.17)} \right\} \tag{2.18}$$

is equivalent to solve the MHE-*i* problems (2.9), in the sense that $\hat{x}^i_{t-N/t}, \{\hat{w}^i_{k/t}\}^t_{k=t-N}$ is a solution to (2.9) if and only if $\hat{\mathbf{x}}_{t-N/t}, \{\hat{\mathbf{w}}_{k/t}\}^t_{k=t-N}$ is a solution to (2.18), where $\hat{\mathbf{w}}_{k/t} = (\hat{w}^1_{k/t}, \dots, \hat{w}^M_{k/t})$.

Let t_1 verify $t-N \leq t_1 \leq t$. We define the *transit cost* of a generic state $z \in \mathbb{R}^n$ at time t_1, computed at instant t as

$$\begin{array}{c} \Xi^i_{t_1/t}(z) = \min_{\hat{x}^i_{t-N}, \{\hat{w}^i_k\}^t_{k=t-N}} \left\{ J^i(t-N, t, \hat{x}^i_{t-N}, \hat{w}^i, \hat{\bar{v}}^i, \Gamma^i_{t-N}) \right. \\ \left. \text{subject to (2.10) and } \hat{x}^i_{t_1} = z \right\} \end{array} \tag{2.19}$$

Note that the associated optimization problem is feasible for all z in $\mathscr{X} = f(\mathbb{X}, \mathbb{W}) \cap \mathbb{X}$ and therefore $\mathscr{X}$ is the domain of $\Xi^i_{t_1/t}(z)$. The collective transit cost of a generic state $\mathbf{x} \in \mathbb{R}^{nM}$ at time t_1, computed at instant t is defined as

$$\begin{array}{c} \boldsymbol{\Xi}_{t_1/t}(\mathbf{x}) = \min_{\hat{\mathbf{x}}_{t-N}, \{\hat{\mathbf{w}}_k\}^t_{k=t-N}} \left\{ \mathbf{J}(t-N, t, \hat{\mathbf{x}}_{t-N}, \hat{\mathbf{w}}, \hat{\bar{\mathbf{v}}}, \boldsymbol{\Gamma}_{t-N}) \right. \\ \left. \text{subject to (2.17) and } \hat{\mathbf{x}}_{t_1} = \mathbf{x} \right\} \end{array} \tag{2.20}$$

and it holds that

$$\boldsymbol{\Xi}_{t_1/t}(\hat{\mathbf{x}}_{t_1}) = \sum_{i=1}^{M} \Xi^i_{t_1/t}(\hat{x}^i_{t_1}) \tag{2.21}$$

In view of Assumption 2 and (2.15a), the cost function $\mathbf{L}$ is continuous, bounded, positive definite, and satisfies the following inequality for all $\mathbf{w} \in \mathbb{R}^{M \cdot m}$, $\bar{\mathbf{v}} \in \mathbb{R}^{\sum_{i=1}^M \bar{p}_i}$

$$\underline{\boldsymbol{\gamma}}_L(\|(\bar{\mathbf{v}}, \mathbf{w})\|) \leq \mathbf{L}(\bar{\mathbf{v}}, \mathbf{w}) \tag{2.22}$$

where $\underline{\boldsymbol{\gamma}}_L \in \mathscr{K}_\infty$.

Furthermore, the initial penalties $\Gamma^{C,i}_{t-N}$ and $\Gamma^{0,i}_{t-N}$ must be defined in order to fulfill the following collective condition.

Assumption 3 *There exists $\underline{\boldsymbol{\gamma}}_0 \in \mathscr{K}_\infty$ such that the following inequalities are verified*

$$\underline{\boldsymbol{\gamma}}_0(\|\mathbf{x} - \mathbf{K}\hat{\mathbf{x}}_{t-N/t-1}\|) \leq \boldsymbol{\Gamma}^C_{t-N}(\mathbf{x}; \mathbf{K}\hat{\mathbf{x}}_{t-N/t-1}), \forall \mathbf{x} \in \mathbb{X}^M \tag{2.23a}$$

$$\underline{\boldsymbol{\gamma}}_0(\|\mathbf{x} - \hat{\mathbf{x}}_{t-N/t-1}\|) \leq \boldsymbol{\Gamma}^0_{t-N}(\mathbf{x}; \hat{\mathbf{x}}_{t-N/t-1}), \forall \mathbf{x} \in \mathbb{X}^M \tag{2.23b}$$

$$\boldsymbol{\Gamma}_{t-N}(\mathbf{z}) \leq \boldsymbol{\Xi}_{t-N/t-1}(\mathbf{z}), \forall \mathbf{z} = \mathbb{1}_M \otimes z, z \in \mathscr{X} \tag{2.23c}$$

Assumption 3 is similar to Assumption 4.17 in [43]. However, there are two key differences. First, inequalities in (2.23c) must hold only for the consensus states $\mathbf{z}$. In particular, we highlight that if $\hat{\mathbf{x}}_{t-N/t-1} = \mathbf{z}$ then $\mathbf{K}\hat{\mathbf{x}}_{t-N/t-1} = \mathbf{z}$ and hence $\boldsymbol{\Theta}^*_{t-1}$ is a global lower bound to $\boldsymbol{\Gamma}_{t-N}$. Second, similarly to [13], as an upper bound to $\boldsymbol{\Gamma}_{t-N}$ we use the transit cost instead of the arrival cost (see Definition 4.16 in [43]).

Note that, guaranteeing that (2.23) is verified is a challenging issue, which is still an open problem in the centralized [43], as well as in the decentralised context. In the special case when (2.1), (2.3) is a linear system, if the stage and initial penalty cost functions are quadratic, as shown in [13, 15], it is possible to provide recursive distributed equations for updating the penalty weighting matrices $\Pi^{C,i}_{t-N}$ and $\Pi^{0,i}_{t-N}$ in order to satisfy Assumption 3, and conditions to guarantee that these matrices remain bounded, in such a way that Assumption 2 is not violated. In the nonlinear context, empirical solutions can be either to compute $\Pi^{C,i}_{t-N}$ and $\Pi^{0,i}_{t-N}$ on the basis of quadratic local approximations of the transit costs or to assign constant values to $\Pi^{C,i}_{t-N}$ and $\Pi^{0,i}_{t-N}$, as it is done in the example shown in Section 2.6.

2.4 Collective stability properties of NDMHE

The main purpose of this section is to extend the stability results of [43] for centralized MHE to the proposed NDMHE scheme.

2.4.1 Collective stability results

Definition 3 *Let Σ be system* (2.1) *with $w = 0$ and denote by $x_\Sigma(t, x_0)$ the state reached by Σ at time t starting from initial condition x_0. Assume that the trajectory $x_\Sigma(t, x_0)$ is feasible,* i.e., $x_\Sigma(t, x_0) \in \mathbb{X}$ *for all t. Define also the collective vectors $\mathbf{x}_0 = \mathbb{1}_M \otimes x_0$ and $\mathbf{x}_\Sigma(t, x_0) = \mathbb{1}_M \otimes x_\Sigma(t, x_0) \in \mathbb{X}^M$. NDMHE is* collectively stable *if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\hat{\mathbf{x}}_{0/N-1} - \mathbf{x}_0\| < \delta$ implies that $\|\hat{\mathbf{x}}_{t-N/t} - \mathbf{x}_\Sigma(t-N, x_0)\| < \varepsilon \; \forall t \geq N$. Also, NDMHE is* collectively asymptotically stable *if it is stable and asymptotically convergent,* i.e.

$$\|\hat{\mathbf{x}}_{t-N/t} - \mathbf{x}_\Sigma(t-N, x_0)\| \stackrel{t\to\infty}{\longrightarrow} 0 \tag{2.24}$$

□

Notice that the condition (2.24) is equivalent to *individual* convergence for all the nodes estimates, *i.e.*

$$\|\hat{x}^i_{t-N/t} - x_\Sigma(t-N, x_0)\| \stackrel{t\to\infty}{\longrightarrow} 0 \tag{2.25}$$

for all $i \in \mathscr{V}$.

Moreover, as in [40], convergence is defined assuming that the model generating the data is noiseless, but the possible presence of noise is taken into account in the state estimation algorithm.
Before to state our main result, we need to introduce the following dynamical system, describing the dynamics of the variable $\boldsymbol{\eta}_t \in \mathbb{R}^{n\cdot M}$

$$\boldsymbol{\eta}_t = \mathbf{P}_{UD}\mathbf{K}\left[\mathbf{f}\left(\tilde{\mathbf{P}}(\boldsymbol{\eta}_{t-1} + \mathbf{P}_D\mathbf{x}_\Sigma(t-N-1, x_0) + \boldsymbol{\alpha}^x_{t-1}), 0\right) + \boldsymbol{\alpha}^w_t\right] + \mathbf{P}_{UD}\boldsymbol{\alpha}^C_t \tag{2.26}$$

where $\boldsymbol{\alpha}^w_t$, $\boldsymbol{\alpha}^x_t$, $\boldsymbol{\alpha}^C_t$ and $\mathbf{x}_\Sigma(t-N-1, x_0)$ are input terms. In the following we resort to the definition of incremental input-to-state stability [6].

Definition 4 *System* (2.26) *is incrementally input-to-state stable (δISS) with respect to the input triplet $(\boldsymbol{\alpha}^x_t, \boldsymbol{\alpha}^w_t, \boldsymbol{\alpha}^C_t)$, if there exist $\beta \in \mathscr{KL}$, $\sigma_\alpha \in \mathscr{K}_\infty$ such that, for any $t \geq 0$, any pair of initial conditions $\boldsymbol{\eta}_{j,0}$, $j = 1,2$ and any pair of input triplets $(\boldsymbol{\alpha}^x_{j,t}, \boldsymbol{\alpha}^w_{j,t}, \boldsymbol{\alpha}^C_{j,t})$, $j = 1,2$ one has*

$$\begin{aligned} \|\boldsymbol{\eta}_{1,t} - \boldsymbol{\eta}_{2,t}\| \leq \beta(\|\boldsymbol{\eta}_{1,0} - \boldsymbol{\eta}_{2,0}\|, t) + \\ +\sigma_\alpha(\|(\boldsymbol{\alpha}^x_{1,k}, \boldsymbol{\alpha}^w_{1,k}, \boldsymbol{\alpha}^C_{1,k}) - (\boldsymbol{\alpha}^x_{2,k}, \boldsymbol{\alpha}^w_{2,k}, \boldsymbol{\alpha}^C_{2,k})\|_{[0:t]}) \end{aligned} \tag{2.27}$$

We are now in the position to state the main result.

Theorem 1 *Under Assumptions 2 and 3, if the system* (2.26) *is δISS with respect to the input triplet $(\boldsymbol{\alpha}^x_t, \boldsymbol{\alpha}^w_t, \boldsymbol{\alpha}^C_t)$ then NDMHE is collectively asymptotically stable.*

Note that, if the system is regionally MHE detectable by any sensor, $\overline{P^i_{UD}} = \mathbf{0}_{n\times n}$ for all $i \in \mathscr{V}$, and hence $\mathbf{P}_{UD} = \mathbf{0}_{nM\times nM}$. Therefore, from equation (2.26) one has that $\boldsymbol{\eta}_t = 0$, and the δISS condition required by Theorem 1 is trivially satisfied.
As shown in the proof of Lemma 2 in the Appendix, the dynamics (2.26) governs the regionally

undetectable components of the state estimates. Therefore δISS of (2.26) implies that the regionally undetectable components of the estimation error vanish when $(\boldsymbol{\alpha}_t^w, \boldsymbol{\alpha}_t^x, \boldsymbol{\alpha}_t^C)$ tends to zero.
For linear constrained systems δISS of (2.26) is implied by the much simpler condition that a suitably defined matrix (matrix Φ in (23) of [14]) is Schur. In the nonlinear context, a sufficient condition is provided in the next section.

2.4.2 A sufficient condition for asymptotic stability of NDMHE

In this section we will provide a sufficient condition which implies the δISS property of (2.26). In the nonlinear context, the system (2.26) can be viewed as the interconnection of M dynamically coupled subsystems. The small gain theorem for networks [10] can be applied for guaranteeing δISS of (2.26) on the basis of the δISS properties of individual sensors and suitable conditions on their interconnections. Similar arguments lead to the result stated in the following theorem.

Theorem 2 *Define $C_{ij} = \overline{P}^i_{UD}\tilde{P}^j\overline{P}^j_{UD}\tilde{P}^j$ and let Γ be a $M \times M$ matrix whose elements γ_{ij} $i, j \in \mathscr{V}$ are $\gamma_{ij} = k_{ij}\|C_{ij}\| l_x^j$ if $i \neq j$, and $\gamma_{ii} = k_{ii} l_x^i$. Assume that one can assign matrix K compatible with the graph $\mathscr{G}$ in such a way that the matrix Γ with elements γ_{ij} is Schur. Then, if Assumption 1 is verified, system* (2.26) *is δISS.*

Notice that, in general, the problem of providing k_{ij}s such that, at the same time (i) K is compatible with the graph $\mathscr{G}$, i.e. it satisfies (2.4), (ii) Γ is Schur, may not have a solution. If this occurs, the δISS property of system (2.26) may be guaranteed by using a “global” analysis approach, see for example [6]. However, for many classes of systems, Theorem 2 provides a powerful design tool for assigning weights k_{ij} even without computing the system’s Lipschitz constants l_x^i, which is not, in general, a trivial task.

For example, consider the case where $\mathscr{V}_D$ is not empty and where, for each node in $\mathscr{V}_{UD}$, there exists an incoming directed path stemming from a regionally MHE detectable node. The latter is satisfied, for instance, when the graph is strongly connected (i.e., there exists a direct path from each node and any other node). In this case it is possible to show that K verifying (i) and (ii) always exists, and that its design can be carried out by the following algorithm.

Algorithm 1

1) for each $i \in \mathscr{V}_D$, set $k_{ii} = 1$;

2) for each $i \in \mathscr{V}_{UD}$, select $k_{ii} = 0$;

3) for each $i \in \mathscr{V}_{UD}$ select a node $j \in \mathscr{V}_D$ and a path from j to i, in such a way that each node in the path has at most one neighbor. We denote with $\mathscr{E}^*$ the set of edges selected in this way;

4) for all edges $(i, j) \in \mathscr{E}^*$, choose $k_{ij} = 1$, while for all edges $(i, j) \in \mathscr{E} \setminus \mathscr{E}^*$, set $k_{ij} = 0$. □

By selecting K according to Algorithm 1 the following result holds.

Corollary 1 *Assume that Assumption 1 holds, that $\mathscr{V}_D$ is non-empty and, for all $i \in \mathscr{V}_{UD}$, there exists an incoming directed path stemming from a node in $\mathscr{V}_D$. Then, if K is selected according to Algorithm 1, system* (2.26) *is δISS.*

Some comments are in order. First, notice that Algorithm 1 is equivalent to Algorithm 1 in [15], which is devised in the context of linear distributed MHE. Interestingly, both algorithms are based on the detectability/observability properties of one or more "leading sensors" (nodes in $\mathcal{V}_D$) and on graph topological properties. The conditions for their application are therefore straightforward to verify, and their application guarantees convergence to zero of the estimation error of the unobservable part of the state, for all sensors (see also Corollary 2 in [15]).

Second, observe that Algorithm 1 implicitly provides a rule for connecting a new regionally MHE detectable/MHE undetectable sensor to the network without spoiling stability of NDMHE, allowing for reconfigurability of the estimation scheme when new sensors are added.

Finally note that, using the same argument as in [15], it is possible to prove that, under the assumptions of Corollary 1, the choice of a matrix K is not unique and the available degrees of freedom in the definition of a suitable K can be used to reduce the conservativeness imposed by Algorithm 1. In fact the generated matrix K is lower triangular, up to a permutation of the node indexes. However, the same results can be obtained by any stochastic matrix $\overline{K}$ compatible with $\mathcal{G}$ with: (a) the same diagonal elements of the matrix K obtained with Algorithm 1; (b) non-zero elements in the lower triangular part; (c) sufficiently small elements in the upper triangular part. This choice allows for a full exploitation of the communication links and hence faster convergence of the estimates to a common value is expected. Moreover, the presence of more links results in an increased reliability against communication faults.

2.5 Generalization

In general, the system (2.1), (2.3) cannot be written in the form (2.6). The general case is the case where we assume that there exists a diffeomorphism $T_i : \mathbb{R}^n \to \mathbb{R}^n$, $T_i^{-1} : x_t \mapsto \xi_t^i = T_i^{-1}(x_t)$ such that, by changing coordinates, and being $\xi_t^i = (\xi_t^{UD,i}, \xi_t^{D,i})$ the state of the equivalent system, one has

$$\xi_{t+1}^{UD,i} = f^{UD,i}(\xi_t^{UD,i}, \xi_t^{D,i}, w_t^i) \tag{2.28a}$$
$$\xi_{t+1}^{D,i} = f^{D,i}(\xi_t^{D,i}, w_t^i) \tag{2.28b}$$
$$\bar{y}_t^i = \tilde{h}^i(\xi_t^{D,i}) + \bar{v}_t^i \tag{2.28c}$$

where the subsystem (2.28b), (2.28c) is MHE detectable. It follows that $\xi_t^{D,i} \in \mathbb{R}^{n_D^i}$ and $\xi_t^{UD,i} \in \mathbb{R}^{n-n_D^i}$ denote regionally MHE detectable and regionally MHE undetectable components of ξ_t^i, respectively.

Let $\overline{P}_{UD}^i$ and $\overline{P}_D^i$ be the $n \times n$ orthonormal projection matrices defined in such a way that $\overline{P}_{UD}^i \xi_t^i = (\xi_t^{UD,i}, 0)$ and $\overline{P}_D^i \xi_t^i = (0, \xi_t^{D,i})$, respectively. Note that, differently from the previous case, here $\overline{P}_{UD}^i$ and $\overline{P}_D^i$ are diagonal matrices. Furthermore, let the map $\mathbf{T} : \mathbb{R}^{M \cdot n} \to \mathbb{R}^{M \cdot n}$ be such that, for $\boldsymbol{\xi}_t = (\xi_t^1, \ldots, \xi_t^M)$, one has $\mathbf{T}(\boldsymbol{\xi}_t) = (T_1(\xi_t^1), \ldots, T_M(\xi_t^M))$.

One can prove that Theorems 1 and 2 and Corollary 1 still hold under the following Lipschitz assumption on T_i and T_i^{-1}.

Assumption 4 *For all $i \in \mathcal{V}$, there exist $\alpha_T, \alpha_{T^{-1}} > 0$ such that*

$$\left\| \left. \frac{\partial T_i(\xi)}{\partial \xi} \right|_{\tilde{\xi}^i} \right\| \le \alpha_T^i \tag{2.29a}$$
$$\left\| \left. \frac{\partial T_i^{-1}(x)}{\partial x} \right|_{\tilde{x}} \right\| \le \alpha_{T^{-1}}^i \tag{2.29b}$$

for $\tilde{\xi}^i = T_i^{-1}(\tilde{x})$, *for all* $\tilde{x} \in \mathbb{R}^n$, *and* $\alpha_T^i \leq \alpha_T$, $\alpha_{T^{-1}}^i \leq \alpha_{T^{-1}}$ $\forall i \in \mathscr{V}$.

In the general case equation (2.26) becomes

$$\boldsymbol{\eta}_t = \mathbf{P}_{UD}\mathbf{T}^{-1}\left\{\mathbf{K}\left[\mathbf{f}\left(\mathbf{T}\left(\boldsymbol{\eta}_{t-1} + \mathbf{P}_D\boldsymbol{\xi}_\Sigma(t-N-1,x_0) + \right.\right.\right.\right.$$
$$\left.\left.\left.\left. + \boldsymbol{\alpha}_{t-1}^{\xi}\right), 0\right) + \boldsymbol{\alpha}_t^w\right] + \boldsymbol{\alpha}_t^C\right\} \tag{2.30}$$

where $\boldsymbol{\xi}_\Sigma(t,x_0) = \mathbf{T}^{-1}(\mathbf{x}_\Sigma(t,x_0))$. For more details see [18].

2.6 Example

Consider the system, composed by four Van der Pol oscillators:

$$\left\{\begin{array}{lcl}
x_{t+1}^1 & = & x_t^1 + 0.06[x_t^2 + (x_t^1 - 0.02(x_t^1)^3)] + k_{cons}(x_t^7 - x_t^1) + w_t^1 \\
x_{t+1}^2 & = & x_t^2 - 0.06x_t^1 + w_t^2 \\
x_{t+1}^3 & = & x_t^3 + 0.072[x_t^4 + 1.1(x_t^3 - 0.015(x_t^3)^3)] + k_{cons}(x_t^1 - x_t^3) + w_t^3 \\
x_{t+1}^4 & = & x_t^4 - 0.072x_t^3 + w_t^4 \\
x_{t+1}^5 & = & x_t^5 + 0.09[x_t^6 + 0.8(x_t^5 - 0.01(x_t^5)^3)] + k_{cons}(x_t^3 - x_t^5) + w_t^5 \\
x_{t+1}^6 & = & x_t^6 - 0.09x_t^5 + w_t^6 \\
x_{t+1}^7 & = & x_t^7 + 0.096[x_t^8 + 0.9(x_t^7 - 0.013(x_t^7)^3)] + k_{cons}(x_t^5 - x_t^7) + w_t^7 \\
x_{t+1}^8 & = & x_t^8 - 0.096x_t^7 + w_t^8
\end{array}\right. \tag{2.31}$$

where we assume that w_t^i is a white noise with uniform distribution in the interval $[-0.5, 0.5]$. Note that, if $k_{cons} = 0.2$ the oscillators are coupled and they are uncoupled if $k_{cons} = 0$. Assume that four sensors are providing measures and are transmitting both estimates and measurements according to the graph in Figure 2.1.

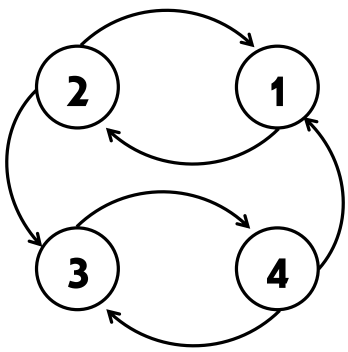

Figure 2.1: Scheme of the network in the example.

The individual sensing models of the four sensors are

$$\begin{array}{lcl}
y_t^1 & = & x_t^2 + v_t^1 \\
y_t^2 & = & x_t^4 + v_t^2 \\
y_t^3 & = & x_t^6 + v_t^3 \\
y_t^4 & = & x_t^8 + v_t^4
\end{array} \tag{2.32}$$

where v_t^i are Gaussian random variable with zero mean and variance equal to $R^i = 1$ for all $i = 1,\ldots,4$. Note that the collective sensing model is

$$\mathbf{y}_t^* = \mathbf{h}^*(x_t) + \mathbf{v}_t = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t^1 \\ x_t^2 \\ \vdots \\ x_t^8 \end{bmatrix} + \begin{bmatrix} v_t^1 \\ v_t^2 \\ v_t^3 \\ v_t^4 \end{bmatrix} \tag{2.33}$$

We can easily verify that the graph is collectively uniformly observable. In fact, the observability map $\mathscr{O}(x_t)$ of the collective system (2.31), (2.33) is

$$\mathscr{O}(x_t) = \begin{bmatrix} \mathbf{h}^*(x_t) \\ \mathbf{h}^*(f(x_t)) \end{bmatrix}$$

Up to a permutation of the rows of $\mathscr{O}(x_t)$, one obtains the matrix

$$\mathscr{O}^*(x_t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.06 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.072 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.09 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.096 & 1 \end{bmatrix} \begin{bmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \\ x_t^4 \\ x_t^5 \\ x_t^6 \\ x_t^7 \\ x_t^8 \end{bmatrix}$$

which is globally invertible for all values of k_{cons}.
To guarantee the incremental input-to-state stability of system (2.26) we will resort to the sufficient condition provided by Theorem 2. In general, $l_x^i \geq 0$ for all $i \in \mathscr{V}$. Note that the regional measurement of sensors 1 and 2 include (y_t^1, y_t^2), and the regional measurement of sensors 3 and 4 include (y_t^3, y_t^4). Therefore, being the system collectively observable (from the collective measurement $(y_t^1, y_t^2, y_t^3, y_t^4)$), it is easy to show that $C_{32} = 0$ and $C_{14} = 0$.
We assign $k_{ii} = 0$ for all $i \in \mathscr{V}$, and $k_{12} = 0$, $k_{14} = 1$, $k_{21} = 1$, $k_{32} = 1$, $k_{34} = 0$, $k_{43} = 1$, resulting in

$$K = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Recall that the non-diagonal entries of Γ, γ_{ij} $i \neq j$ are equal to $\gamma_{ij} = k_{ij}\|C_{ij}\|l_x^j$. In view of the previous choice of K, we obtain that

$$\Gamma = \begin{bmatrix} 0 & 0 & 0 & \|C_{14}\|l_x^4 \\ \|C_{21}\|l_x^1 & 0 & 0 & 0 \\ 0 & \|C_{32}\|l_x^2 & 0 & 0 \\ 0 & 0 & \|C_{43}\|l_x^3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \|C_{21}\|l_x^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \|C_{43}\|l_x^3 & 0 \end{bmatrix}$$

which is Schur for all values of $\|C_{21}\|l_x^1$ and $\|C_{43}\|l_x^3$. Furthermore, Assumption 1 is verified for $\mathbb{X}$ bounded such that all the state trajectories are contained in $\mathbb{X}$ and therefore Theorem 2 guarantees that system (2.26) is δISS.

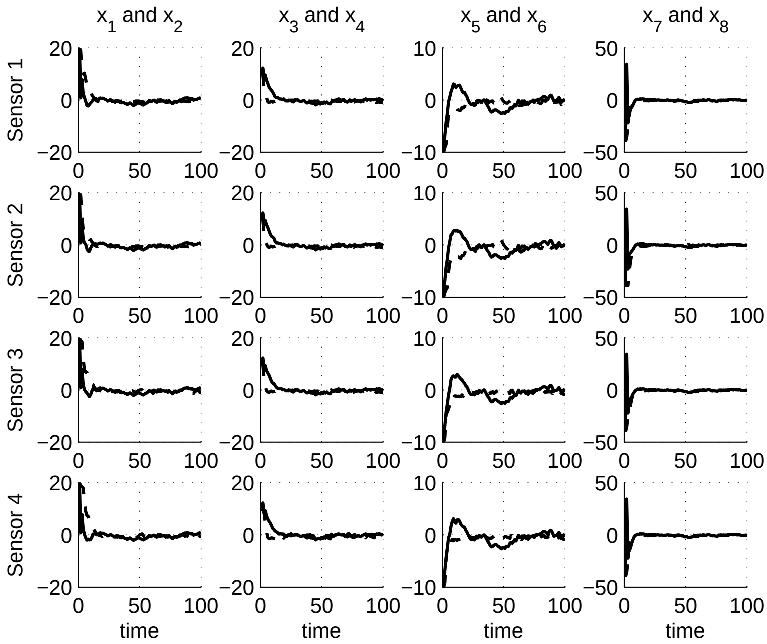

Figure 2.2: Components of the estimation error $x_t - \hat{x}^i_{t/t}$ of the different sensors, with $k_{cons} = 0.2$. The components with even index (corresponding to the states x^2_t, x^4_t, x^6_t, x^8_t) are depicted with dashed lines, and the components with odd index (corresponding to the states x^1_t, x^3_t, x^5_t and x^7_t) are depicted with solid lines.

We set $N = 2$, $L^i(v^i_t, w^i_t) = \frac{1}{2}\|w^i_t\|_{(Q^i)^{-1}} + \frac{1}{2}\|v^i_t\|_{(R^i)^{-1}}$ where $Q^i = I_8$, $R^i = 1$ and the regularization term weight and the consensus term weight are equal to $\Pi^{O,i}_{t-N} = I_8$ and $\Pi^{C,i}_{t-N} = 0.2 \cdot I_8$ for all $i = 1, \dots, 4$, respectively.

The estimation errors produced by all sensors are shown, with $k_{cons} = 0.2$ and with $k_{cons} = 0$, in Figure 2.2 and in Figure 2.3, respectively. Notice that, in both cases, the estimation errors of the states which are directly observed by each sensor converge to zero very fast, while the estimation errors of the states which are not directly observed by each sensor asymptotically tend to zero thanks to the consensus action embodied in the NDMHE scheme.

2.7 Proofs

Proof of Theorem 1

Under (2.23) the first step towards the convergence of the NDMHE estimator is the following lemma.

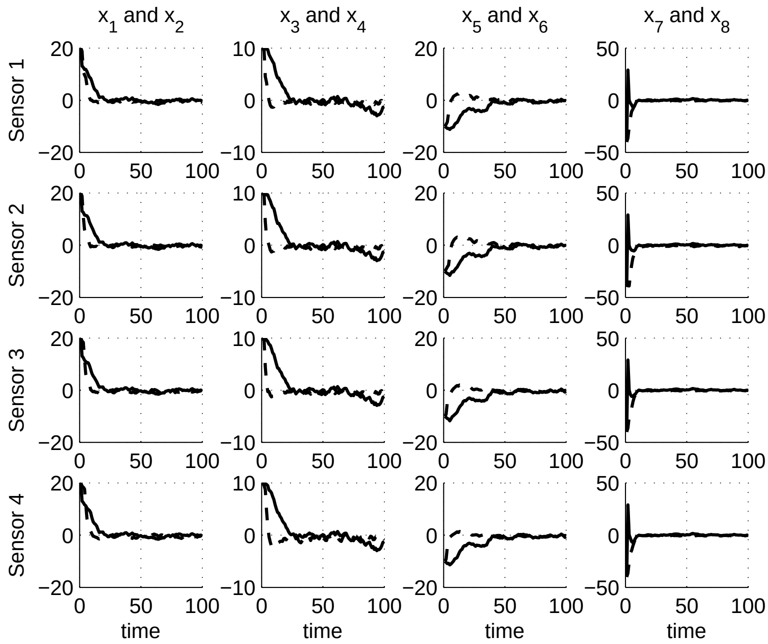

Figure 2.3: Components of the estimation error $x_t - \hat{x}^i_{t/t}$ of the different sensors, with $k_{cons} = 0$. The components with even index (corresponding to the states x_t^2, x_t^4, x_t^6, x_t^8) are depicted with dashed lines, and the components with odd index (corresponding to the states x_t^1, x_t^3, x_t^5 and x_t^7) are depicted with solid lines.

Lemma 1 *If Assumption 3 holds then*

$$\sum_{k=t-N}^{t} \mathbf{L}(\hat{\mathbf{v}}_{k/t}, \hat{\mathbf{w}}_{k/t}) \xrightarrow{t\to\infty} 0 \tag{2.34a}$$

$$\underline{\boldsymbol{\gamma}}_0(\|\hat{\mathbf{x}}_{t-N/t} - \mathbf{K}\hat{\mathbf{x}}_{t-N/t-1}\|) \xrightarrow{t\to\infty} 0 \tag{2.34b}$$

$$\underline{\boldsymbol{\gamma}}_0(\|\hat{\mathbf{x}}_{t-N/t} - \hat{\mathbf{x}}_{t-N/t-1}\|) \xrightarrow{t\to\infty} 0 \tag{2.34c}$$

For the sake of clarity the proof of Theorem 1 is split in the proof of the next two lemmas.

Lemma 2 *Under Assumptions 2 and 3, if the system* (2.26) *is δISS with respect to the input triplet* $(\boldsymbol{\alpha}_t^x, \boldsymbol{\alpha}_t^w, \boldsymbol{\alpha}_t^C)$ *then NDMHE is asymptotically convergent.*

Finally, the following lemma deals with collective stability of the estimates.

Lemma 3 *Under the assumptions of Lemma 2 then NDMHE is collectively stable.*

Proof of Lemma 1
The proof is similar to the one in Proposition 5 in [40]. For all $t \geq 0$ one has, from (2.23)

$$\begin{aligned} \boldsymbol{\Theta}_t^* - \boldsymbol{\Theta}_{t-1}^* \quad \geq \quad & \textstyle\sum_{k=t-N}^{t} \mathbf{L}(\hat{\mathbf{v}}_{k/t}, \hat{\mathbf{w}}_{k/t}) + \underline{\boldsymbol{\gamma}}_0(\|\hat{\mathbf{x}}_{t-N/t} - \mathbf{K}\hat{\mathbf{x}}_{t-N/t-1}\|) + \\ & + \underline{\boldsymbol{\gamma}}_0(\|\hat{\mathbf{x}}_{t-N/t} - \hat{\mathbf{x}}_{t-N/t-1}\|) \end{aligned} \tag{2.35}$$

showing that the sequence $\mathbf{\Theta}_t^*$ is increasing. Note that since $x_\Sigma(t,x_0)\in\mathbb{X}$, the transit cost $\mathbf{\Xi}_{t-N/t-1}(\mathbf{x}_\Sigma(t-N,x_0))$ is well defined, $\forall t\geq N$, i.e., $x_\Sigma(t,x_0)\in\mathscr{X}$.
By optimality $\mathbf{\Theta}_t^*\leq\mathbf{\Xi}_{t-N+1/t}(\mathbf{x}_\Sigma(t-N+1,x_0))$, $\forall t$. Moreover, from Definition 3

$$\mathbf{\Xi}_{t-N+1/t}(\mathbf{x}_\Sigma(t-N+1,x_0))\leq\mathbf{J}(t-N,t,\mathbf{x}_\Sigma(t-N,x_0),0,0,\mathbf{\Gamma}_{t-N})$$

From (2.16), one has

$$\mathbf{J}(t-N,t,\mathbf{x}_\Sigma(t-N,x_0),0,0,\mathbf{\Gamma}_{t-N})=\mathbf{\Gamma}_{t-N}(\mathbf{x}_\Sigma(t-N,x_0))$$

and in view of (2.23)

$$\mathbf{\Theta}_t^*\leq\mathbf{\Xi}_{t-N/t-1}(\mathbf{x}_\Sigma(t-N,x_0))$$

We can iterate this procedure and prove that

$$\mathbf{\Theta}_t^*\leq\mathbf{\Gamma}_0(\mathbf{x}_0) \tag{2.36}$$

for all t, for any $\mathbf{x}_0\in\mathbb{X}^M$. From (2.36) and (2.13b)-(2.13c), the sequence $\mathbf{\Theta}_t^*$ is bounded. Therefore, the sequence $\mathbf{\Theta}_t^*$ converges and from (2.35) the equations (2.34a) and (2.34b) and (2.34c) follow. ■

Proof of Lemma 2

Next we create, for each sensor node i, a single estimate sequence $\bar{x}_t^i=\hat{x}_{t/t+N}^i$ by concatenating MHE sequences for the system (2.1). This gives the state sequences $\bar{x}_k^i$ and the corresponding augmented disturbance sequences $\bar{\bar{w}}_k^i=(\bar{w}_k^{i,1},\bar{w}_k^{i,2})$

$$\bar{x}_{t+1}^i=f(\bar{x}_t^i,\bar{w}_t^{i,1})+\bar{w}_t^{i,2} \tag{2.37a}$$
$$\bar{y}_t^i=h(\bar{x}_t^i)+\hat{\bar{v}}_{t/t+N} \tag{2.37b}$$

where

$$\bar{w}_t^{i,1}=\hat{w}_{t/t+N}^i \tag{2.38a}$$
$$\bar{w}_t^{i,2}=\hat{x}_{t+1/t+N+1}^i-\hat{x}_{t+1/t+N}^i \tag{2.38b}$$

Define a sequence

$$\alpha_t^{x,i}=\bar{P}_D^i\left(\bar{x}_{t-N}^i-x_\Sigma(t-N,x_0)\right) \tag{2.39}$$

In view of (2.6) and Definition 1, one has

$$\|\alpha_t^{x,i}\|\leq\beta_D(\|\bar{x}_0^i-x_0\|,t-N)+\gamma_1(\|\bar{\bar{w}}_k^i\|_{[0:t-N-1]})+\gamma_2(\|\hat{\bar{v}}_{k/k+N}^i\|_{[0:t-N]}) \tag{2.40a}$$

This, according to (2.38), implies that there exist functions $\gamma_{11},\gamma_{12}\in\mathscr{K}$ such that

$$\begin{aligned}\|\alpha_t^{x,i}\|\leq&\beta_D(\|\bar{x}_0^i-x_0\|,t-N)+\\&+\gamma_{11}(\|\hat{w}_{k/k+N}^i\|_{[0:t-N-1]})+\gamma_2(\|\hat{\bar{v}}_{k/k+N}^i\|_{[0:t-N]})+\\&+\gamma_{12}(\|\hat{x}_{k+1/k+N+1}^i-\hat{x}_{k+1/k+N}^i\|_{[0:t-N-1]})\end{aligned} \tag{2.40b}$$

We define $\mathbf{x}_\Sigma(t,x_0) = \mathbb{1}_M \otimes x_\Sigma(t,x_0)$, $\overline{\mathbf{x}}_t = (\overline{x}_t^1, \ldots, \overline{x}_t^M)$ and $\boldsymbol{\alpha}_t^x = (\alpha_t^{x,1}, \ldots, \alpha_t^{x,M})$. Collectively (2.39) results in

$$\boldsymbol{\alpha}_t^x = \mathbf{P}_D\left(\overline{\mathbf{x}}_{t-N} - \mathbf{x}_\Sigma(t-N,x_0)\right) \tag{2.41}$$

Furthermore, applying the mean value theorem for vector functions (see Appendix A in [43]), we can write

$$\hat{x}^i_{k+1/t} = f(\hat{x}^i_{k/t}, 0) + \alpha^{w,i}_{k/t} \tag{2.42}$$

where

$$\|\alpha^{w,i}_{k/t}\| \leq l\|\hat{w}^i_{k/t}\| \tag{2.43}$$

l being the Lipschitz constant in (2.2a). We define $\boldsymbol{\alpha}_t^w = (\alpha^{w,1}_{t-N-1/t-1}, \ldots, \alpha^{w,M}_{t-N-1/t-1})$ and $\mathbf{f}(\hat{\mathbf{x}}_k) = (f(\hat{x}^1_k), \ldots, f(\hat{x}^M_k))$. Collectively we write (2.42) as

$$\hat{\mathbf{x}}_{t-N/t-1} = \mathbf{f}(\hat{\mathbf{x}}_{t-N-1/t-1}, 0) + \boldsymbol{\alpha}_t^w \tag{2.44}$$

From Lemma 1, (2.34a) holds and together with (2.22) one has that $\|(\hat{\mathbf{w}}_{t-N/t}, \hat{\mathbf{v}}_{t-N/t})\| \to 0$ as $t \to +\infty$. Hence, $\|(\hat{w}^i_{t-N/t}, \hat{v}^i_{t-N/t})\| \to 0$ as $t \to +\infty$ for all $i \in \mathscr{V}$. Similarly, from (2.34c), $\|\hat{x}^i_{t+1/t+N+1} - \hat{x}^i_{t+1/t+N}\| \to 0$ as $t \to +\infty$ for all $i \in \mathscr{V}$. In view of (2.40b) and Proposition 4.2 in [43] (convergence of the state under δIOSS, see Definition 1), this implies that $\alpha_t^{x,i} \to 0$ as $t \to +\infty$, i.e.

$$\overline{P}^i_D\left(\overline{x}^i_{t-N} - x^i_\Sigma(t-N,x_0)\right) \overset{t\to+\infty}{\longrightarrow} 0 \ \forall i \in \mathscr{V} \tag{2.45}$$

Moreover

$$\boldsymbol{\alpha}_t^w \overset{t\to+\infty}{\longrightarrow} 0 \tag{2.46}$$

Finally, from (2.34b) and (2.23), we obtain that

$$\hat{\mathbf{x}}_{t-N/t} = \mathbf{K}\hat{\mathbf{x}}_{t-N/t-1} + \boldsymbol{\alpha}_t^C \tag{2.47}$$

where $\boldsymbol{\alpha}_t^C \to 0$ as $t \to +\infty$.

According to (2.44) and (2.47) we can write

$$\begin{aligned}\mathbf{P}_{UD}\overline{\mathbf{x}}_{t-N} &= \mathbf{P}_{UD}\left(\mathbf{K}\hat{\mathbf{x}}_{t-N/t-1} + \boldsymbol{\alpha}_t^C\right) \\ &= \mathbf{P}_{UD}\left[\mathbf{K}\left(\mathbf{f}(\overline{\mathbf{x}}_{t-N-1}, 0) + \boldsymbol{\alpha}_t^w\right) + \boldsymbol{\alpha}_t^C\right]\end{aligned}$$

where, using (2.41) we can write

$$\overline{\mathbf{x}}_{t-N-1} = \tilde{\mathbf{P}}(\mathbf{P}_D\mathbf{x}_\Sigma(t-N-1,x_0) + \mathbf{P}_{UD}\overline{\mathbf{x}}_{t-N-1} + \boldsymbol{\alpha}^x_{t-1})$$

Hence, we obtain that the dynamics of $\mathbf{P}_{UD}\overline{\mathbf{x}}_{t-N}$ evolves according to (2.26). By direct calculation the dynamics of variable $\mathbf{P}_{UD}\mathbf{x}_\Sigma(t-N,x_0)$ is given by (2.26), with $\boldsymbol{\alpha}_t^w, \boldsymbol{\alpha}_t^x, \boldsymbol{\alpha}_t^C = 0$. We define $\hat{\boldsymbol{\eta}}_t = \mathbf{P}_{UD}\overline{\mathbf{x}}_{t-N}$ and $\boldsymbol{\eta}_\Sigma(t,x_0) = \mathbf{P}_{UD}\mathbf{x}_\Sigma(t-N,x_0)$. We introduce the function $\mathbf{F}$, so that the dynamical equations for $\hat{\boldsymbol{\eta}}_t$ and $\boldsymbol{\eta}_\Sigma(t,x_0)$ can be written as

$$\hat{\boldsymbol{\eta}}_t = \mathbf{F}(\hat{\boldsymbol{\eta}}_{t-1}, \mathbf{P}_D\mathbf{x}_\Sigma(t-N-1,x_0), \boldsymbol{\alpha}^x_{t-1}, \boldsymbol{\alpha}^w_t, \boldsymbol{\alpha}^C_t) \tag{2.48a}$$

$$\boldsymbol{\eta}_\Sigma(t,x_0) = \mathbf{F}(\boldsymbol{\eta}_\Sigma(t-1,x_0), \mathbf{P}_D\mathbf{x}_\Sigma(t-N-1,x_0), 0, 0, 0) \tag{2.48b}$$

According to Definition 4, if the system (2.48a) is δISS, then there exist $\beta \in \mathcal{KL}$, $\sigma_\alpha \in \mathcal{K}_\infty$ such that

$$\|\hat{\boldsymbol{\eta}}_t - \boldsymbol{\eta}_\Sigma(t,x_0)\| \leq \beta(\|\hat{\boldsymbol{\eta}}_0 - \boldsymbol{\eta}_\Sigma(0,x_0)\|,t) + \sigma_\alpha(\|(\boldsymbol{\alpha}_k^x, \boldsymbol{\alpha}_k^w, \boldsymbol{\alpha}_k^C)\|_{[0:t]}) \tag{2.49}$$

If (2.49) holds then, using Proposition 4.2 in [43] one has $\|\hat{\boldsymbol{\eta}}_t - \boldsymbol{\eta}_\Sigma(t,x_0)\| \to 0$ because $(\boldsymbol{\alpha}_t^x, \boldsymbol{\alpha}_t^w, \boldsymbol{\alpha}_t^C) \to 0$ as $t \to +\infty$. This, together with (2.46) implies that $\hat{\mathbf{x}}_{t-N/t} \to \hat{\mathbf{x}}_\Sigma(t-N,x_0)$ as $t \to +\infty$. ∎

Proof of Lemma 3

For the sake of clarity, the proof is split in four parts.

1) We choose $\rho > 0$ such that

$$\sum_{k=t-N}^{t} \mathbf{L}(\hat{\bar{\mathbf{v}}}_{k/t}, \hat{\mathbf{w}}_{k/t}) < \rho \tag{2.50a}$$

$$\boldsymbol{\Gamma}_{t-N}^C(\hat{\mathbf{x}}_{t-N/t}; \mathbf{K}\hat{\mathbf{x}}_{t-N/t-1}) < \rho \tag{2.50b}$$

$$\boldsymbol{\Gamma}_{t-N}^0(\hat{\mathbf{x}}_{t-N/t}; \hat{\mathbf{x}}_{t-N/t-1}) < \rho \tag{2.50c}$$

From (2.50a), in view of (2.22), we obtain that $\|\hat{\mathbf{w}}_{k/t}\| < \underline{\boldsymbol{\gamma}}_L^{-1}(\rho)$ and $\|\hat{\bar{\mathbf{v}}}_{k/t}\| < \underline{\boldsymbol{\gamma}}_L^{-1}(\rho)$. Furthermore, from (2.50c) and in view of (2.23), $\|\hat{\mathbf{x}}_{t-N/t} - \hat{\mathbf{x}}_{t-N/t-1}\| \leq \underline{\boldsymbol{\gamma}}_0^{-1}(\rho)$. Then, from (2.40b) there exist $\boldsymbol{\gamma}_3 \in \mathcal{K}$, $\boldsymbol{\beta}_D \in \mathcal{KL}$ such that $\boldsymbol{\alpha}_t^x$ in (2.41) satisfies

$$\|\boldsymbol{\alpha}_t^x\| < \boldsymbol{\gamma}_3(\rho) + \boldsymbol{\beta}_D(\|\bar{\mathbf{x}}_0 - \mathbf{x}_0\|, 0) \tag{2.51}$$

Similarly, from (2.43), we obtain that there exists a function $\tilde{\boldsymbol{\gamma}}_w \in \mathcal{K}$ such that $\boldsymbol{\alpha}_t^w$ satisfies

$$\|\boldsymbol{\alpha}_t^w\| < \tilde{\boldsymbol{\gamma}}_w(\rho) \tag{2.52}$$

Furthermore, from (2.50b), in view of Assumption 3 and equation (2.47), we obtain that $\boldsymbol{\alpha}_t^C$ satisfies

$$\|\boldsymbol{\alpha}_t^C\| < \underline{\boldsymbol{\gamma}}_0^{-1}(\rho) \tag{2.53}$$

2) In view of (2.41)

$$\begin{aligned}\|\mathbf{P}_D(\bar{\mathbf{x}}_{t-N} - \mathbf{x}_\Sigma(t-N,x_0)))\| &< \boldsymbol{\gamma}_3(\rho) + \\ &+ \boldsymbol{\beta}_D(\|\bar{\mathbf{x}}_0 - \mathbf{x}_\Sigma(0,x_0)\|, 0)\end{aligned} \tag{2.54}$$

Note that, in view of (2.47)

$$\begin{aligned}\|\bar{\mathbf{x}}_0 - \mathbf{x}_\Sigma(0,x_0)\| &= \|\mathbf{K}\hat{\mathbf{x}}_{0/N-1} + \boldsymbol{\alpha}_N^C - \mathbf{x}_0\| \\ &\leq \|\mathbf{K}\hat{\mathbf{x}}_{0/N-1} - \mathbf{x}_0\| + \|\boldsymbol{\alpha}_N^C\|\end{aligned} \tag{2.55a}$$

From (2.53) there exist functions $\boldsymbol{\beta}_{D,x}, \boldsymbol{\gamma}_3^* \in \mathcal{K}$ such that

$$\begin{aligned}\|\mathbf{P}_D(\bar{\mathbf{x}}_{t-N} - \mathbf{x}_\Sigma(t-N,x_0))\| &< \boldsymbol{\gamma}_3^*(\rho) + \\ &+ \boldsymbol{\beta}_{D,x}(\|\hat{\mathbf{x}}_{0/N-1} - \mathbf{x}_0\|)\end{aligned} \tag{2.56}$$

Furthermore, in view of (2.49), recalling that $\hat{\boldsymbol{\eta}}_t = \mathbf{P}_{UD}\overline{\mathbf{x}}_{t-N}$ and $\boldsymbol{\eta}_\Sigma(t,x_0) = \mathbf{P}_{UD}\mathbf{x}_\Sigma(t-N,x_0)$, one has

$$\begin{aligned} &\|\mathbf{P}_{UD}\left(\overline{\mathbf{x}}_{t-N} - \mathbf{x}_\Sigma(t-N,x_0)\right)\| \leq \\ &\leq \beta\left(\|\mathbf{P}_{UD}\left(\overline{\mathbf{x}}_0 - \mathbf{x}_\Sigma(0,x_0)\right)\|,t\right)+ \\ &+\sigma_\alpha(\|(\boldsymbol{\alpha}^x_k,\boldsymbol{\alpha}^w_k,\boldsymbol{\alpha}^C_k)\|_{[0:t]}) \end{aligned} \tag{2.57}$$

Note that

$$\|\mathbf{P}_{UD}\left(\overline{\mathbf{x}}_0 - \mathbf{x}_\Sigma(0,x_0)\right)\| \leq \|\overline{\mathbf{x}}_0 - \mathbf{x}_\Sigma(0,x_0)\|$$

From (2.53), (2.49) and (2.55a) one concludes that

$$\begin{aligned} &\|\mathbf{P}_{UD}\left(\overline{\mathbf{x}}_{t-N} - \mathbf{x}_\Sigma(t-N,x_0)\right)\| \leq \\ &\leq \beta(2\|\mathbf{K}(\hat{\mathbf{x}}_{0/N-1} - \mathbf{x}_0)\|,0) + \beta(2\|\boldsymbol{\alpha}^C_N\|,0)+ \\ &+\sigma_\alpha(\|(\boldsymbol{\alpha}^x_k,\boldsymbol{\alpha}^w_k,\boldsymbol{\alpha}^C_k)\|_{\mathscr{L}_\infty}) \end{aligned} \tag{2.58}$$

and so, there exist a functions $\tilde{\sigma}_\alpha \in \mathscr{K}_\infty$ such that

$$\begin{aligned} &\|\mathbf{P}_{UD}\left(\overline{\mathbf{x}}_{t-N} - \mathbf{x}_\Sigma(t-N,x_0)\right)\| \leq \\ &\leq \beta(2\|\mathbf{K}(\hat{\mathbf{x}}_{0/N-1} - \mathbf{x}_0)\|,0) + \tilde{\sigma}_\alpha(\rho) \end{aligned} \tag{2.59}$$

3) From equation (2.36) it follows that, for all $t > 0$, $\boldsymbol{\Theta}^*_t \leq \boldsymbol{\Gamma}_0(\mathbf{x}_0) = \boldsymbol{\Gamma}_0(\mathbf{K}\mathbf{x}_0)$, where the last equality holds because $\mathbf{x}_0 = \mathbb{1}_M \otimes x_0$. If we define $\delta = \hat{\mathbf{x}}_{0/N-1} - \mathbf{x}_0$, then there exists $\boldsymbol{\gamma}_0 \in \mathscr{K}_\infty$ such that, in view of (2.13b)

$$\sum_{k=t-N}^{t} \mathbf{L}(\hat{\overline{\mathbf{v}}}_{k/t},\hat{\mathbf{w}}_{k/t}) < \boldsymbol{\gamma}_0(\delta) \tag{2.60a}$$

$$\boldsymbol{\Gamma}^C_{t-N}(\hat{\mathbf{x}}_{t-N/t};\mathbf{K}\hat{\mathbf{x}}_{t-N/t-1}) < \boldsymbol{\gamma}_0(\delta) \tag{2.60b}$$

$$\boldsymbol{\Gamma}^0_{t-N}(\hat{\mathbf{x}}_{t-N/t};\hat{\mathbf{x}}_{t-N/t-1}) < \boldsymbol{\gamma}_0(\delta) \tag{2.60c}$$

We define $\delta = \boldsymbol{\gamma}_0^{-1}(\rho)$, the function $\boldsymbol{\gamma}_0^{-1}$ being a $\mathscr{K}_\infty$ function as well, see [41]. From equation (2.59) one concludes that there exists a function $\boldsymbol{\gamma}_4 \in \mathscr{K}_\infty$ such that

$$\|\mathbf{P}_{UD}\left(\overline{\mathbf{x}}_{t-N} - \mathbf{x}_\Sigma(t-N,x_0)\right)\| \leq \boldsymbol{\gamma}_4(\rho) \tag{2.61}$$

From (2.56) and (2.61), we obtain that there exists a function $\boldsymbol{\gamma}_5 \in \mathscr{K}_\infty$ such that

$$\|\hat{\mathbf{x}}_{t-N/t} - \mathbf{x}_\Sigma(t-N,x_0)\| < \boldsymbol{\gamma}_5(\rho)$$

Let $\varepsilon = \boldsymbol{\gamma}_5(\rho)$, one can compute $\delta = \boldsymbol{\gamma}_0^{-1}\left(\boldsymbol{\gamma}_5^{-1}(\varepsilon)\right)$ and the condition for collective stability is verified.

■

Proof of Theorem 2

System (2.26) can be viewed as the interconnection of M subsystems having η_t^i as state variables, $i = 1,\dots M$, with $\boldsymbol{\eta}_t = (\eta_t^1,\dots,\eta_t^M)$. The dynamics of η_t^i is described by the subsystem

$$\begin{aligned}\eta_t^i &= \bar{P}_{UD}^i\left\{\sum_{j=1}^M k_{ij}\left[f\left(\tilde{P}^j\left(\eta_{t-1}^j+\bar{P}_D^j x_\Sigma(t-N-1,x_0)+\alpha_{t-1}^{x,j}\right),0\right)+\alpha_t^{w,j}\right]+\alpha_t^{C,i}\right\}\\ &= \bar{P}_{UD}^i\left\{k_{ii}\left[f\left(\tilde{P}^i\left(\eta_{t-1}^i+\bar{P}_D^i x_\Sigma(t-N-1,x_0)+\alpha_{t-1}^{x,i}\right),0\right)+\alpha_t^{w,i}\right]+\sum_{j\neq i}k_{ij}f\left(x_\Sigma(t-N-1;x_0),0\right)+\right.\\ &\left.+\sum_{j\neq i}k_{ij}\left[f\left(\tilde{P}^j\left(\eta_{t-1}^j+\bar{P}_D^j x_\Sigma(t-N-1,x_0)+\alpha_{t-1}^{x,j}\right),0\right)-f\left(x_\Sigma(t-N-1;x_0),0\right)+\alpha_t^{w,j}\right]+\alpha_t^{C,i}\right\}\end{aligned} \tag{2.62}$$

Recall that $\eta_\Sigma^i(t,x_0)$ is the solution to system (2.62) with inputs $\eta_{t-1}^j = \bar{P}_{UD}^j x_\Sigma(t-N-1,x_0)$, $j\neq i$, and with zero input terms $\alpha_t^{x,j}$, $\alpha_t^{w,j}$, $\alpha_t^{C,j}$. Denote $\Delta\eta_t^i = \eta_t^i - \eta_\Sigma^i(t,x_0)$ for all $i = 1,\dots,M$. It results that the dynamics of $\Delta\eta_t^i$ is given by the system

$$\begin{aligned}\Delta\eta_t^i = \bar{P}_{UD}^i&\left\{k_{ii}\left[f\left(x_\Sigma(t-N-1,x_0)+\tilde{P}^i\left(\Delta\eta_{t-1}^i+\alpha_{t-1}^{x,i}\right),0\right)-f\left(x_\Sigma(t-N-1;x_0),0\right)+\alpha_t^{w,i}\right]+\right.\\ &\left.+\sum_{j\neq i}k_{ij}\left[f\left(x_\Sigma(t-N-1,x_0)+\tilde{P}^j\left(\Delta\eta_{t-1}^j+\alpha_{t-1}^{x,j}\right),0\right)-f\left(x_\Sigma(t-N-1;x_0),0\right)+\alpha_t^{w,j}\right]+\alpha_t^{C,i}\right\}\end{aligned} \tag{2.63}$$

For simplicity, we denote $\tilde{\alpha}_t^i = \sum_{j=1}^M k_{ij}\bar{P}_{UD}^i(\alpha_t^{w,j}+\alpha_t^{C,i})$ and

$$g_{ij}(\Delta\eta_{t-1}^j; x_\Sigma(t-N-1),\alpha_{t-1}^{x,j}) = \tag{2.64}$$

$$\bar{P}_{UD}^i\left[f\left(x_\Sigma(t-N-1,x_0)+\tilde{P}^j\left(\Delta\eta_{t-1}^j+\alpha_{t-1}^{x,j}\right),0\right)-f\left(x_\Sigma(t-N-1;x_0),0\right)\right] \tag{2.65}$$

in such a way that

$$\begin{aligned}&\Delta\eta_t^i = \phi_i(\Delta\eta_t^i,\{\Delta\eta_t^j\}_{j\neq i},\alpha_{t-1}^{x,i},\tilde{\alpha}_t^i) =\\ &k_{ii}g_{ii}(\Delta\eta_{t-1}^i;x_\Sigma(t-N-1),\alpha_{t-1}^{x,i})+\sum_{j\neq i}k_{ij}g_{ij}(\Delta\eta_{t-1}^j;x_\Sigma(t-N-1),\alpha_{t-1}^{x,j})+\tilde{\alpha}_t^i\end{aligned} \tag{2.66}$$

We have shown that δISS of (2.26) is equivalent to ISS from $\boldsymbol{\alpha}_t = (\alpha_t^1,\dots,\alpha_t^M)$ and from $\boldsymbol{\alpha}_t^x$ to $\Delta\boldsymbol{\eta}_t = (\Delta\eta_t^1,\dots,\Delta\eta_t^M)$ for the collection of systems (2.66).

By resorting to the Lipschitz Assumption 1 on the function f, functions g_{ij} in (2.65) can be analyzed more into details. For brevity we write $\tilde{f}(x) = f(x,0)$. For notational simplicity we will denote (when clear from the context) x_Σ as a short-hand for $x_\Sigma(t-N-1,x_0)$. In view of the mean value theorem

$$\tilde{f}(x_\Sigma+\tilde{P}^j(\Delta\eta_{t-1}^j+\alpha_{t-1}^{x,j})) = \tilde{f}(x_\Sigma+\tilde{P}^j\Delta\eta_{t-1}^j)+\Phi(x_\Sigma+\tilde{P}^j\Delta\eta_{t-1}^j, x_\Sigma+\tilde{P}^j(\Delta\eta_{t-1}^j+\alpha_{t-1}^{x,j}))\tilde{P}^j\alpha_{t-1}^{x,j} \tag{2.67}$$

where $\Phi(x_1,x_2) = \int_0^1 \frac{\partial\tilde{f}(x)}{\partial x}|_{x_1+(x_2-x_1)s}ds$. In view of Assumption 1, $\Phi(x_\Sigma+\tilde{P}^j(\Delta\eta_{t-1}^j), x_\Sigma+\tilde{P}^j(\Delta\eta_{t-1}^j+\alpha_{t-1}^{x,j}))\tilde{P}^j\alpha_{t-1}^{x,j}$ is bounded if $\alpha_{t-1}^{x,j}$ is bounded. Furthermore

$$\begin{aligned}\tilde{f}(x_\Sigma+\tilde{P}^j\Delta\eta_{t-1}^j)-\tilde{f}(x_\Sigma) &= \tilde{P}^j\left(\begin{bmatrix}f^{UD,j}(x_\Sigma^{UD}+\Delta\eta_{t-1}^{UD,j},x_\Sigma^D,0)\\ f^{D,j}(x_\Sigma^D,0)\end{bmatrix}-\begin{bmatrix}f^{UD,j}(x_\Sigma^{UD},x_\Sigma^D,0)\\ f^{D,j}(x_\Sigma^D,0)\end{bmatrix}\right)\\ &= \tilde{P}^j\begin{bmatrix}f^{UD,j}(x_\Sigma^{UD}+\Delta\eta_{t-1}^{UD,j},x_\Sigma^D,0)-f^{UD,j}(x_\Sigma^{UD},x_\Sigma^D,0)\\ 0\end{bmatrix}\end{aligned} \tag{2.68}$$

By further applying the mean value theorem we obtain that

$$\begin{bmatrix} f^{UD,j}(x_\Sigma^{UD}+\Delta\eta_{t-1}^{UD,j},x_\Sigma^D,0)-f^{UD,j}(x_\Sigma^{UD},x_\Sigma^D,0)\\ 0\end{bmatrix}=\Phi^{UD,j}(x_\Sigma,x_\Sigma+\Delta\eta_{t-1}^j)\Delta\eta_{t-1}^j \quad (2.69)$$

where

$$\Phi^{UD,j}(x_1,x_2)=\begin{bmatrix}\int_0^1 \frac{\partial f^{UD,j}(x^{UD},x^D)}{\partial x^{UD}}|_{x_1+(x_2-x_1)s}ds & 0\\ 0 & 0\end{bmatrix}$$

In the sequel, for notational simplicity, if possible and when clear from the context, function arguments will be omitted. In view of its structure, note that $\Phi^{UD,j}=\bar{P}_{UD}^j\tilde{P}^j\Phi^{UD,j}$ for all $j\in\mathscr{V}$. Let us define $C_{ij}=\bar{P}_{UD}^i\tilde{P}^j\bar{P}_{UD}^j\tilde{P}^j$. In view of (2.67), (2.68) and (2.69) we rewrite (2.66) as

$$\Delta\eta_t^i = k_{ii}\Phi^{UD,i}\Delta\eta_{t-1}^i+\sum_{j\neq i}k_{ij}\bar{P}_{UD}^i\tilde{P}^j\Phi^{UD,j}\Delta\eta_{t-1}^j+\bar{\alpha}_t^i = k_{ii}\Phi^{UD,i}\Delta\eta_{t-1}^i+\sum_{j\neq i}k_{ij}C_{ij}\Phi^{UD,j}\Delta\eta_{t-1}^j+\bar{\alpha}_t^i \quad (2.70)$$

where $\bar{\alpha}_t^i$ is bounded if $\alpha_{t-1}^{x,j}$ is bounded. From (2.70) we obtain that

$$\|\Delta\eta_t^i\| \leq k_{ii}l_x^i\|\Delta\eta_{t-1}^i\|+\sum_{j\neq i}k_{ij}\|C_{ij}\|l_x^j\|\Delta\eta_{t-1}^j\|+\|\bar{\alpha}_t^i\| \quad (2.71)$$

Now define the matrix Γ as specified. We write $\mathbf{e}_t=(\|\Delta\eta_t^1\|,\ldots,\|\Delta\eta_t^M\|)$ and $\bar{\boldsymbol{\alpha}}_t=(\|\bar{\alpha}_t^1\|,\ldots,\|\bar{\alpha}_t^M\|)$. From (2.71), $\mathbf{e}_t\leq\Gamma\mathbf{e}_{t-1}+\bar{\boldsymbol{\alpha}}_t$ element-wise. Define the sequence $\mathbf{e}_t^*$ such that $\mathbf{e}_0^*=\mathbf{e}_0$, and

$$\mathbf{e}_t^*=\Gamma\mathbf{e}_{t-1}^*+\bar{\boldsymbol{\alpha}}_t \quad (2.72)$$

Since Γ is a positive matrix then, for all $t\geq 0$, $0\leq\mathbf{e}_t\leq\mathbf{e}_t^*$. Then, if Γ is Schur, the system (2.72) is ISS from $\bar{\boldsymbol{\alpha}}_t$ to $\mathbf{e}_t^*$. Therefore the overall system is ISS from $(\alpha_{t-1}^{x,i},\tilde{\alpha}_t^i)$ to $\Delta\eta_t^i$, $i\in\mathscr{V}$. ■

Proof of Corollary 1

First notice that, by assigning K according to Algorithm 1, K is compatible with $\mathscr{G}$ and $k_{ii}l_x^i=0$ for all $i\in\mathscr{V}$. In fact, for all $i\in\mathscr{V}_D$, $l_x^i=0$ and, for all $i\in\mathscr{V}_{UD}$, $k_{ii}=0$ according to step 2 of Algorithm 1. From the graph $(\mathscr{V},\mathscr{E})$, we derive a subgraph $\mathscr{G}^*=(\mathscr{V},\mathscr{E}^*)$, by selecting edges $(i,j)\in\mathscr{E}^*\subseteq\mathscr{E}$ according to step 3 of Algorithm 1. By construction, the graph $\mathscr{G}^*$ is a maximal forest [7], i.e. a graph composed by a number of mutually disjoint trees and covering all the nodes. Moreover, the root of each tree is a regionally MHE detectable node while all other nodes are regionally MHE undetectable. It follows that each row of the matrix K produced by Algorithm 1 has only one off-diagonal element that is different from zero.

Up to a permutation of the node indexes, K is lower triangular. It follows that the matrix $\Gamma=\{\gamma_{ij}\}$ enjoys the following properties: (i) its diagonal entries are equal to zero, (ii) it is lower triangular. Therefore Γ is Schur. Finally, resorting to Theorem 2, it is possible to prove the δISS of (2.26). ■

Chapter 3

Partition-based moving horizon estimation for nonlinear systems

3.1 Introduction

Many industrial processes and physical systems are composed by a large number of interconnected units, each one described by a dynamic model. In these cases, the computational load associated to the design of a unique centralized controller can be high; moreover, a centralized approach does not take advantage of the sparsity of the system. For these reasons, research in the design of distributed control systems, in particular with Model Predictive Control (MPC), has produced many significant results and is going to play an ever increasing role, see e.g. the results achieved in he HD-MPC project, the review [46] and the references therein. However, most of the distributed control methods proposed so far are state-feedback, so that in order to guarantee a fully distributed control design, also distributed state estimation algorithms dealing with constraints are needed. The availability of distributed state-estimation algorithms is of paramount importance in many different engineering applications, such as process control [51], power networks [48] and transport networks [45].

Early works in distributed estimation were aimed at reducing the computational complexity of centralized Kalman filters by parallelizing computations, see e.g. [22, 37], under the assumption that each subsystem has full knowledge of the whole dynamics. Subsequently, [30] proposed a solution based on the use of reduced-order and decoupled models for each subsystem, while subsystems with overlapping states were considered in [25, 50, 49, 51]. While the estimation schemes proposed in [51] require an all-to-all communication among subsystems, in [25, 50, 49] the topology of the network is defined by dependencies among the states of the subsystems resulting in a fully distributed scheme. More recently, distributed state estimators for sensor networks where each sensor measures just some of the system outputs and computes the estimate of the overall state have been studied in [5, 8, 24, 34]. In these methods, convergence of the estimates to a common value is achieved through consensus algorithms under weak assumptions on the topology of the communication network. Along the same lines and in order to cope with constraint on noise and state variables, in the HD-MPC project distributed MHE algorithms for sensor networks have been proposed, see deliverable D5.2 and [13, 15, 16] .

Three partition-based MHE algorithms (PMHE) for linear constrained systems decomposed into interconnected subsystems without overlapping states have also been developed in the HD-MPC project and described in [17]. In these algorithms, which differ in terms of communication requirements, accuracy and computational complexity, each subsystem solves a reduced-order MHE problem in order to estimate its own states based on the estimate of the other subsystems' states transmitted by its neighbors.

The results of [17] are here extended to the case of nonlinear systems so as to cope with the majority of problems arising in process control, where the nonlinear dynamic phenomena have often to be considered in order to guarantee the accuracy of the solution. The convergence properties of the method are investigated and sufficient conditions are given. These conditions turn out to be automatically satisfied when the directed graph describing interconnections among subsystems is acyclic.
The proposed partition-based MHE is applied to the problem of estimating the levels and flow rates in the model of three cascade river reaches, which represent a significant part of the Hydro Power Valley benchmark extensively studied in the project (see the workpackages WP6 and WP7). Interconnections between successive reaches are due to the dependence of the input flow rate of the downstream reaches to the level of the final section of the upstream ones, which cannot be measured, but just estimated from the available measures collected along the reach.

The chapter is structured as follows. Section 3.2 introduces nonlinear partitioned systems and the main assumptions concerning their dynamics. Section 3.3 describes the proposed MHE algorithm, while convergence results are provided in Section 3.4. The illustrative example is considered in Section 3.5. For the sake of readability, the proofs of the main results are collected at the end of the chapter.

Notation. In the chapter, the following notation is adopted. I_n and 0 denote the $n \times n$ identity matrix and the matrix of zero elements whose dimensions will be clear from the context. The notation $\|z\|_S^2$ stands for $z^T Sz$, where S is a symmetric positive-semidefinite matrix. Given a set of scalar variables $v = \{v^{i_1}, \dots, v^{i_n}\}$, $i_1 < i_2 < \cdots < i_n$, we use the short-hand (v) or $(v^{i_1}, \dots, v^{i_n})$ to denote the vector $[v^{i_1}, \dots, v^{i_n}]^T$. By extension, if $v^{i_1}, \dots, v^{i_n}$ are sets of scalar variables, (v) or $(v^{i_1}, \dots, v^{i_n})$ denote $((v^{i_1}), \dots, (v^{i_n}))$. With a little abuse of notation, and when clear from the context, we will use v instead of (v) i.e., identify sets of variables with the corresponding column vectors. Given a function $f(x_1, \dots, x_n) : \mathbb{D} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ we define $\arg(f) = \{x_i : f \text{ is not constant in } x_i \text{ on } \mathbb{D}\}$.

For a discrete-time signal w_t and $a, b \in \mathbb{N}$, $a \leq b$, we denote $(w_a, w_{a+1}, \dots, w_b)$ with $w_{[a:b]}$. For the definition of positive-definite, $\mathscr{K}$, $\mathscr{K}_\infty$ and $\mathscr{KL}$ functions we defer the reader to [43]. Finally, given $x_k, x_k^* \in \mathbb{R}^n$, we define $\|x_k - x_k^*\|_{[a:b]} = \max_{k \in [a:b]} \|x_k - x_k^*\|$, where $\|\cdot\|$ denotes the Euclidean norm.

3.2 Nonlinear large-scale systems

Consider the discrete-time nonlinear system

$$\mathbf{x}_{t+1} = \mathbf{f}(\mathbf{x}_t, \mathbf{u}_t) + \mathbf{w}_t, \tag{3.1}$$

where $\mathbf{x}_t = (x_t^1, \dots, x_t^n) \in \mathbb{R}^n$ is the state, $\mathbf{w}_t \in \mathbb{R}^n$ is the process noise and $\mathbf{u}_t \in \mathbb{R}^m$ is the input. Measurements are performed according to the sensing model

$$\mathbf{y}_t = \mathbf{h}(\mathbf{x}_t, \mathbf{u}_t) + \mathbf{v}_t \tag{3.2}$$

where $\mathbf{v}_t \in \mathbb{R}^p$ is the measurement noise.

We assume that $\mathbf{f}(\mathbf{x}_t, \mathbf{u}_t)$ has continuous partial derivatives with respect to the argument $\mathbf{x}_t$ and also satisfies the following Assumption.

Assumption 5 *Function* $\mathbf{f}$ *is globally Lipschitz with respect to* $\mathbf{x}$. i.e., $\exists l_x > 0 : \forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{X}$, *for all* $\mathbf{u} \in \mathbb{R}^m$ *and*

$$\|\mathbf{f}(\mathbf{x}_1, \mathbf{u}) - \mathbf{f}(\mathbf{x}_2, \mathbf{u})\| \leq l_x \|\mathbf{x}_1 - \mathbf{x}_2\| \tag{3.3}$$

We split system (3.1) into M interconnected submodels by choosing a time-invariant partition of elements of $\mathbf{x}_t$ into the sets $x_t^{[1]}, \ldots, x_t^{[M]}$, $M \le n$ so that $\mathbf{x}_t = (x_t^{[1]}, \ldots, x_t^{[M]})$ up to an index permutation, where $x_t^{[i]} \in \mathbb{R}^{n_i}$ for all $i = 1, \ldots, M$. Accordingly, we define $\mathbf{f}(\mathbf{x}_t, \mathbf{u}_t) = (f^{[1]}(\mathbf{x}_t, \mathbf{u}_t), \ldots, f^{[M]}(\mathbf{x}_t, \mathbf{u}_t))$, and $\mathbf{w}_t = (w_t^{[1]}, \ldots, w_t^{[M]})$ so that the dynamics of subsystem i is

$$x_{t+1}^{[i]} = f^{[i]}(\mathbf{x}_t, \mathbf{u}_t) + w_t^{[i]} \tag{3.4}$$

where $w_t^{[i]} \in \mathbb{R}^{n_i}$ for all $i = 1, \ldots, M$. The partition induces an interconnection network in the form of a directed graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ where nodes in $\mathscr{V} = \{1, \ldots, M\}$ are subsystems and $(j,i) \in \mathscr{E}$ if and only if $i \neq j$ and $\exists x_t^k \in \arg(f^{[i]}(\cdot, \mathbf{u}_t)) : x_t^k \in x_t^{[j]}$.

Defining $\mathscr{N}_i = \{j : (j,i) \in \mathscr{E}\}$, $u_t^{[i],x} = \{x_t^{[j]}, j \in \mathscr{N}_i\}$ and $u_t^{[i]} = \{u_t^k : u_t^k \in \arg(f^{[i]}(\mathbf{x}_t, \cdot))\} \in \mathbb{R}^{m_i}$ model (3.4) can be written as

$$x_{t+1}^{[i]} = f^{[i]}(x_t^{[i]}, u_t^{[i],x}, u_t^{[i]}) + w_t^{[i]} \tag{3.5}$$

We assume that the state of subsystem i fulfills the constraint $x_t^{[i]} \in \mathbb{X}_i$, where $\mathbb{X}_i$ is a convex set. If $\mathbb{X} = \mathbb{R}^n$ we say that the system is unconstrained and if $M = 1$ we say that the partition is trivial.

In view of Assumption 5 we have that, for all $i = 1, \ldots, M$, there exist Lipschitz constants $l_{ij} > 0 :$ $\forall x_1^{[j]}, x_2^{[j]} \in \mathbb{X}_j$, $j \in \mathscr{N}_i \cup \{i\}$, for all $u^{[i]} \in \mathbb{R}^{m_i}$, such that

$$\begin{aligned} &\| f^{[i]}(x_1^{[i]}, u_1^{[i],x}, u^{[i]}) - f^{[i]}(x_2^{[i]}, u_2^{[i],x}, u^{[i]}) \| \\ &\qquad \le l_{ii} \| x_1^{[i]} - x_2^{[i]} \| + \sum_{j \in \mathscr{N}_i} l_{ij} \| x_1^{[j]} - x_2^{[j]} \| \end{aligned} \tag{3.6}$$

We denote as $\mathscr{L}$ the matrix whose i-th row, $i = 1, \ldots, M$, is composed by the elements l_{ij} if $j \in \mathscr{N}_i \cup \{i\}$ and 0 otherwise. Note that we can interpret matrix $\mathscr{L}$ in (3.6) as a weighted adjacency matrix for the graph (more specifically, the elements a_{ij} $i, j = 1, \ldots, M$ of the adjacency matrix $\mathscr{A}$ verify $a_{ij} = 1$ if $l_{ji} > 0$ and $a_{ij} = 0$ otherwise).

As for the outputs of subsystems, we assume that the sets $y_t^{[1]}, \ldots, y_t^{[M]}$ are a time-invariant partition of variables in $\mathbf{y}_t$ and, analogously, the sets $v_t^{[1]}, \ldots, v_t^{[M]}$ are a time-invariant partition of variables in $\mathbf{v}_t$, so that

$$y_t^{[i]} = h^{[i]}(x_t^{[i]}, u_t^{[i]}) + v_t^{[i]} \tag{3.7}$$

for suitable functions $h^{[i]}$. Note that (3.7), besides excluding the case of outputs shared by multiple subsystems, also assumes that $y_t^{[i]}$ only depends upon the local variables $x_t^{[i]}$ and $u_t^{[i]}$. We highlight that these structural assumptions are made only for the sake of simplicity and the main results can be generalized to the case where they do not hold.

From now on, we assume that the system partitioning and the input sequence $\{\mathbf{u}_k\}$ are such that the following observability assumption on the local subsystems is satisfied (see [43] for $M = 1$).

Assumption 6 *For all $i = 1, \ldots, M$ and $j \in \mathscr{N}_i$, there exists $n_i^o \in \mathbb{N} \backslash \{0\}$ and functions $\gamma_w^{[i]}(\cdot)$, $\gamma_{ij}(\cdot)$, $\gamma_y^{[i]}(\cdot) \in \mathscr{K}$ such that for every initial states $x_0^{[i]}$, $x_0^{*[i]}$, for any feasible sequences $x_{[0:k-1]}^{[j]}$, $x_{[0:k-1]}^{*[j]}$ and for any disturbance sequences $w_{[0:k-1]}^{[i]}$, $w_{[0:k-1]}^{*[i]}$*

$$\begin{aligned} \|x_0^{[i]} - x_0^{*[i]}\| \le \gamma_w^{[i]}(\|w_t^{[i]} - w_t^{*[i]}\|_{[0:k-1]}) \\ + \gamma_y^{[i]}(\|y_t^{[i]} - y_t^{*[i]}\|_{[0:k]}) + \textstyle\sum_{j \in \mathscr{N}_i} \gamma_{ij}(\|x_t^{[j]} - x_t^{*[j]}\|_{[0:k-1]}) \end{aligned} \tag{3.8}$$

where $k \geq n_i^o$, and $y_t^{[i]}$ and $y_t^{*[i]}$ are the output sequences stemming from $(w_t^{[i]}, u_t^{[i],x}, u_t^{[i]})$ and $(w_t^{*[i]}, u_t^{*[i],x}, u_t^{[i]})$ with initial conditions $x_0^{[i]}$ and $x_0^{*[i]}$, respectively.

3.3 A non-iterative moving horizon partition-based algorithm

Our aim is to design, for each subsystem, a non-iterative distributed estimation scheme based on neighbor-to-neighbor communication for computing a reliable estimate $\hat{x}^{[i]}$ of $x^{[i]}$ based on the measurements $y^{[i]}$ and on the crosstalk terms $u^{[i],x}$ provided by the estimators associated to the other subsystems. Specifically, by extending to nonlinear systems the results described in [17], we propose a moving horizon estimation (MHE) scheme, which is denoted NPMHE.

3.3.1 Model for estimation and information transmission graph

We denote with $\hat{x}_{t_1/t_2}^{[i]}$ the estimate of $x_{t_1}^{[i]}$ performed at time t_2 by subsystem i and we define $\hat{\mathbf{x}}_{t_1/t_2} = (\hat{x}_{t_1/t_2}^{[1]}, \ldots, \hat{x}_{t_1/t_2}^{[M]})$. At each instant t we assume that an estimate of the crosstalk term $u_k^{[i],x}$ $k = t - N, \ldots, t$ is provided by the neighbors at time $t-1$, therefore allowing for decentralization of the state estimation algorithm. At time t the estimation model is, for $k = t-N, \ldots, t-1$

$$\hat{x}_{k+1}^{[i]} = f^{[i]}(\hat{x}_k^{[i]}, \hat{u}_{k/t-1}^{[i],x}, u_k^{[i]}) + \hat{w}_k^{[i]} \tag{3.9a}$$

$$y_k^{[i]} = h^{[i]}(\hat{x}_k^{[i]}, u_k^{[i]}) + \hat{v}_k^{[i]} \tag{3.9b}$$

and defines constraints of the NPMHE estimation problem given below.

In (3.9), $\hat{u}_{k/t-1}^{[i],x}$ is the set of variables $\{\hat{x}_{k/t-1}^{[j]}, j \in \mathcal{N}_i\}$, denoting the estimates of the system's states available to subsystem i's neighbors at time t. Therefore, subsystems communicate over a network that has the same topology of the interconnection graph and, if $(j,i) \in \mathscr{E}$, then $\hat{x}_{k/t-1}^{[j]}$ for $k = t-N, \ldots, t-1$ are transmitted to subsystem i.

Finally, note that the noise estimates $\hat{w}_k^{[i]}$ and $\hat{v}_k^{[i]}$ in (3.9) encompass both the noise appearing in the equations (3.5), (3.7) and the estimation error due to the uncertainty on $\hat{u}_{k/t-1}^{[i],x}$.

3.3.2 The NPMHE estimation problems

Given an estimation horizon $N \geq 1$, in order to perform the NPMHE algorithm, each node $i \in \mathscr{V}$ at time t solves the constrained minimization problem NMHE-i defined as

$$\Theta_t^{*[i]} = \min_{\hat{x}_{t-N}^{[i]}, \hat{w}^{[i]}} J^{[i]}(t-N, t, \hat{x}_{t-N}^{[i]}, \hat{w}^{[i]}, \hat{v}^{[i]}, \Gamma_{t-N}^{[i]}) \tag{3.10}$$

where $\hat{w}^{[i]}$ and $\hat{v}^{[i]}$ stand for $\hat{w}_{[t-N:t]}^{[i]}$ and $\hat{v}_{[t-N:t]}^{[i]}$, respectively, under the constraints imposed by system (3.9) and

$$\hat{x}_k^{[i]} \in \mathbb{X}_i, \, k = t-N, \ldots, t \tag{3.11}$$

The local cost function $J^{[i]}$ is given by

$$J^{[i]}(t-N,t,\hat{x}^{[i]}_{t-N},\hat{w}^{[i]},\hat{v}^{[i]},\Gamma^{[i]}_{t-N}) = \sum_{k=t-N}^{t} L^{[i]}(\hat{w}^{[i]}_k,\hat{v}^{[i]}_k) + \Gamma^{[i]}_{t-N}(\hat{x}^{[i]}_{t-N};\hat{x}^{[i]}_{t-N/t-1}) \tag{3.12}$$

In (3.12), the functions $L^{[i]}$ and $\Gamma^{[i]}_{t-N}$ are the *stage cost* and the *initial penalty*, respectively, and must be defined in order to satisfy the following assumption.

Assumption 7 *$L^{[i]}$ and $\Gamma^{[i]}_{t-N}$ are continuous, bounded, positive definite and satisfy the following inequalities for all $w^{[i]} \in \mathbb{R}^{n_i}$, $v^{[i]} \in \mathbb{R}^{p_i}$, $x^{[i]}_1, x^{[i]}_2 \in \mathbb{R}^{n_i}$*

$$\underline{\gamma}_L(\|(w^{[i]},v^{[i]})\|) \le L^{[i]}(w^{[i]},v^{[i]}) \tag{3.13a}$$

$$\Gamma^{[i]}_0(x^{[i]}_1;x^{[i]}_2) \le \gamma_0(\|x^{[i]}_1 - x^{[i]}_2\|) \tag{3.13b}$$

where $\underline{\gamma}_L$ and γ_0 are suitable $\mathscr{K}_\infty$ functions.

The quantities $\hat{x}^{[i]}_{t-N/t}$ and $\left\{\hat{w}^{[i]}_{k/t}\right\}_{k=t-N}^{t}$ are the optimizers to (3.10) and $\hat{x}^{[i]}_{k/t}$, $k = t-N+1,\dots,t$ is the local state sequence stemming from $\hat{x}^{[i]}_{t-N/t}$, $\{\hat{u}^{[i]}_{k/t-1}\}_{k=t-N}^{t-1}$ and $\left\{\hat{w}^{[i]}_{k/t}\right\}_{k=t-N}^{t-1}$.

3.3.3 The collective minimization problem

Denote by $\mathbf{J}$ the sum of the local cost functions $J^{[i]}$, given by (3.12), i.e.

$$\mathbf{J} = \sum_{i=1}^{M} J^{[i]}(t-N,t,\hat{x}^{[i]}_{t-N},\hat{w}^{[i]},\hat{v}^{[i]},\Gamma^{[i]}_{t-N}) \tag{3.14}$$

Define the collective vectors $\hat{\mathbf{x}}_t = (\hat{x}^{[1]}_t,\dots,\hat{x}^{[M]}_t) \in \mathbb{R}^n$, $\hat{\mathbf{v}}_t = (\hat{v}^{[1]}_t,\dots,\hat{v}^{[M]}_t) \in \mathbb{R}^p$, $\hat{\mathbf{w}}_t = (\hat{w}^{[1]}_t,\dots,\hat{w}^{[M]}_t) \in \mathbb{R}^n$ and rewrite $\mathbf{J}$ as

$$\mathbf{J} = \sum_{k=t-N}^{t} \mathbf{L}(\hat{\mathbf{w}}_k,\hat{\mathbf{v}}_k) + \boldsymbol{\Gamma}_{t-N}(\hat{\mathbf{x}}_{t-N};\hat{\mathbf{x}}_{t-N/t-1}) \tag{3.15}$$

where $\mathbf{L}$ and $\boldsymbol{\Gamma}_{t-N}$ are given by

$$\mathbf{L}(\hat{\mathbf{w}}_k,\hat{\mathbf{v}}_k) = \sum_{i=1}^{M} L^{[i]}(\hat{w}^{[i]}_k,\hat{v}^{[i]}_k) \tag{3.16a}$$

$$\boldsymbol{\Gamma}_{t-N}(\hat{\mathbf{x}}_{t-N};\hat{\mathbf{x}}_{t-N/t-1}) = \sum_{i=1}^{M} \Gamma^{[i]}_{t-N}(\hat{x}^{[i]}_{t-N};\hat{x}^{[i]}_{t-N/t-1}) \tag{3.16b}$$

We define the function

$$\tilde{\mathbf{f}}(\hat{\mathbf{x}}_k,\hat{\mathbf{x}}_{k/t-1},\mathbf{u}_k) = (f^{[1]}(\hat{x}^{[1]}_k,\hat{u}^{[1],x}_{k/t-1},u^{[1]}_k), \dots, f^{[M]}(\hat{x}^{[M]}_k,\hat{u}^{[M],x}_{k/t-1},u^{[M]}_k))$$

so that constraints (3.9) and (3.11) can be written in the collective form

$$\hat{\mathbf{x}}_{k+1} = \tilde{\mathbf{f}}(\hat{\mathbf{x}}_k, \hat{\mathbf{x}}_{k/t-1}, \mathbf{u}_k) + \hat{\mathbf{w}}_k \tag{3.17a}$$

$$\mathbf{y}_k = \mathbf{h}(\hat{\mathbf{x}}_k, \mathbf{u}_k) + \hat{\mathbf{v}}_k \tag{3.17b}$$

$$\hat{\mathbf{x}}_k \in \mathbb{X} \tag{3.17c}$$

with $k = t-N, \dots, t$. The solution to

$$\min_{\hat{\mathbf{x}}_{t-N}, \hat{\mathbf{w}}} \mathbf{J}(t-N, t, \hat{\mathbf{x}}_{t-N}, \hat{\mathbf{w}}, \hat{\mathbf{v}}, \mathbf{\Gamma}_{t-N}) \tag{3.18}$$

where $\hat{\mathbf{w}}$ and $\hat{\mathbf{v}}$ are short-hand notation for $\hat{\mathbf{w}}_{[t-N:t]}$ and $\hat{\mathbf{v}}_{[t-N:t]}$ respectively, is equivalent to the solution to the MHE-i problems (3.10), in the sense that $\hat{x}^{[i]}_{t-N/t}, \{\hat{w}^{[i]}_{k/t}\}^t_{k=t-N}$ is a solution to (3.10) if and only if $\hat{\mathbf{x}}_{t-N/t}, \{\hat{\mathbf{w}}_{k/t}\}^t_{k=t-N}$ is a solution to (3.18), where $\hat{\mathbf{w}}_{k/t} = (\hat{w}^{[1]}_{k/t}, \dots, \hat{w}^{[M]}_{k/t})$. In fact, at time t, variables $\hat{\mathbf{x}}_{k/t-1}$ are fixed inputs for the system (3.17).
We define the transit cost for subsystem i as

$$\Xi^{[i]}_{[t-N+1:t]/t}(z^{[i]}_{[t-N+1:t]}) = \min_{\hat{x}^i_{t-N}, \hat{w}^i} \Big\{ J^{[i]}(t-N, t, \hat{x}^{[i]}_{t-N}, \hat{w}^{[i]}, \hat{v}^{[i]}, \Gamma^{[i]}_{t-N})$$
$$\text{subject to (3.9), (3.11) and } \hat{x}^{[i]}_k = z^{[i]}_k \text{ for } k = t-N+1, \dots, t \Big\} \tag{3.19}$$

The collective transit cost in a generic sequence $\mathbf{z}_k = (z^{[1]}_k, \dots, z^{[M]}_k) \in \mathbb{R}^n$ $k = t-N+1, \dots, t$, is defined as

$$\mathbf{\Xi}_{[t-N+1:t]/t}(\mathbf{z}_{[t-N+1:t]}) = \min_{\hat{\mathbf{x}}_{t-N}, \hat{\mathbf{w}}} \{ \mathbf{J}(t-N, t, \hat{\mathbf{x}}_{t-N}, \hat{\mathbf{w}}, \hat{\mathbf{v}}, \mathbf{\Gamma}_{t-N})$$
$$\text{subject to (3.17) and } \hat{\mathbf{x}}_k = \mathbf{z}_k \text{ for } k = t-N+1, \dots, t \} \tag{3.20}$$

and it holds that

$$\mathbf{\Xi}_{[t-N+1:t]/t}(\mathbf{z}_{[t-N+1:t]}) = \sum_{i=1}^{M} \Xi^{[i]}_{[t-N+1:t]/t}(z^{[i]}_{[t-N+1:t]}) \tag{3.21}$$

From (3.16) and in view of Assumption 7 there exist suitable $\mathscr{K}_\infty$ functions $\underline{\gamma}^*_L$ and γ^*_0 such that

$$\underline{\gamma}^*_L(\|(\mathbf{w}, \mathbf{v})\|) \leq \mathbf{L}(\mathbf{w}, \mathbf{v}) \tag{3.22a}$$

$$\mathbf{\Gamma}_0(\mathbf{x}_1; \mathbf{x}_2) \leq \gamma^*_0(\|\mathbf{x}_1 - \mathbf{x}_2\|) \tag{3.22b}$$

for all $\mathbf{w} \in \mathbb{R}^n$, $\mathbf{v} \in \mathbb{R}^p$ and $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$.

We discuss now the choice of the initial penalties $\Gamma^{[i]}_{t-N}$. As it will clear in the next section, for convergence of the NPMHE scheme they must fulfill the following assumption.

Assumption 8 *Given a state sequence* $\mathbf{z}_k \in \mathbb{X}$, $k = t-N, \dots, t-1$, *the following inequalities are verified*

$$\mathbf{\Theta}^*_{t-1} \leq \mathbf{\Gamma}_{t-N}(\mathbf{z}_{t-N}; \hat{\mathbf{x}}_{t-N/t-1}) \tag{3.23}$$

$$\sum_{k=t-N}^{t-1} \mathbf{L}(\tilde{\mathbf{f}}(\mathbf{z}_k, \mathbf{z}_k, \mathbf{u}_k) - \tilde{\mathbf{f}}(\mathbf{z}_k, \hat{\mathbf{x}}_{k/t-1}, \mathbf{u}_k), 0) + \tag{3.24}$$
$$+ \mathbf{\Gamma}_{t-N}(\mathbf{z}_{t-N}; \hat{\mathbf{x}}_{t-N/t-1}) \leq \mathbf{\Xi}_{[t-N:t-1]/t-1}(\mathbf{z}_{[t-N:t-1]})$$

where $\mathbf{\Theta}^*_{t-1} = \sum_{i=1}^{M} \Theta^{*[i]}_{t-1}$.

Remark 1 *Note that Assumption 8 is the most critical one. Currently, even in case of a trivial partition where one has that $\sum_{k=t-N}^{t-1}\mathbf{L}(\tilde{\mathbf{f}}(\mathbf{z}_k,\mathbf{z}_k,\mathbf{u}_k)-\tilde{\mathbf{f}}(\mathbf{z}_k,\hat{\mathbf{x}}_{k/t-1},\mathbf{u}_k),0)=0$, there are just approximate methods for computing initial penalties $\mathbf{\Gamma}_{t-N}\neq\mathbf{\Theta}^*_{t-1}$ verifying* (3.23) *and* (3.24)*, see [43].*
Note however that $\mathbf{z}_{[t-N:t-1]}=\hat{\mathbf{x}}_{[t-N:t-1]/t-1}$ are minimizers of $\mathbf{\Xi}_{[t-N:t-1]/t-1}(\mathbf{z}_{[t-N:t-1]})$ and yield $\mathbf{\Xi}_{[t-N:t-1]/t-1}(\mathbf{z}_{[t-N:t-1]})=\mathbf{\Theta}^*_{t-1}$. Hence (3.23)-(3.24) *imply that $\hat{\mathbf{x}}_{[t-N:t-1]/t-1}$ is a minimizer of $\mathbf{\Gamma}_{t-N}$ and that $\mathbf{\Gamma}_{t-N}(\hat{\mathbf{x}}_{t-N/t-1};\hat{\mathbf{x}}_{t-N/t-1})=\mathbf{\Theta}^*_{t-1}$.*

As in [17], a choice for $L^{[i]}$ and $\Gamma^{[i]}_{t-N}$ fulfilling Assumption 7 and (3.23) is to consider the quadratic functions

$$L^{[i]} = \frac{1}{2}\|\hat{w}^{[i]}_k\|^2_{(\mathscr{Q}^{[i]})^{-1}}+\frac{1}{2}\|\hat{v}^{[i]}_k\|^2_{(\mathscr{R}^{[i]})^{-1}} \tag{3.25}$$

$$\Gamma^{[i]}_{t-N} = \frac{1}{2}\|\hat{x}^{[i]}_{t-N}-\hat{x}^{[i]}_{t-N/t-1}\|^2_{(\Pi^{[i]}_{t-N/t-1})^{-1}}+\Theta^{*[i]}_{t-1} \tag{3.26}$$

where $\mathscr{Q}^{[i]}$ and $\mathscr{R}^{[i]}$ are suitable symmetric and positive-definite matrices, and $\Pi^{[i]}_{t-N/t-1}$ is a symmetric semi definite-positive matrix.

It is easy to prove that, under Assumption 5 and if the stage cost and the initial penalty are defined as in (3.25) and (3.26), respectively, Assumption 8 is verified if, for each subsystem i, the following inequality is satisfied, for all sequences $z^{[i]}_k\in\mathbb{X}_i$, $k=t-N,\dots,t-1$

$$\begin{aligned}\frac{1}{2}\sum_{k=t-N}^{t-1} q_i\|z^{[i]}_k-\hat{x}^{[i]}_{k/t-1}\|^2+\frac{1}{2}\|z^{[i]}_{t-N}-\hat{x}^{[i]}_{t-N/t-1}\|^2_{(\Pi^{[i]}_{t-N/t-1})^{-1}}\\ \leq \Xi^{[i]}_{[t-N:t-1]/t-1}(z^{[i]}_{[t-N:t-1]})\end{aligned} \tag{3.27}$$

where, for all i, $q_i=\sum_{j=1}^{M}\tilde{l}_{ji}$, $\tilde{l}_{ij}=2\,l^2_{ij}\sigma_{max}\left((\mathscr{Q}^{[i]})^{-1}\right)$ if $j\in\mathscr{N}_i$ and $\tilde{l}_{ij}=0$ otherwise, and $\sigma_{max}(\cdot)$ denotes the maximum singular value of its argument.
Note that (3.27) is indeed a local re-formulation of Assumption 8 and recall that also the cost (3.12) and the constraints (3.9) depend only upon local variables. In view of this, the implementation of the NPMHE estimation scheme results to be completely decentralized.

3.4 Convergence properties of the proposed estimators

In this section the convergence results reported in [41] for centralized estimators (corresponding to the trivial partition) are extended to the proposed NPMHE scheme. Similarly to [41], these properties are analyzed in a deterministic setting.

Definition 5 *Let Σ be system* (3.1) *with $\mathbf{w}_t=0$ and denote by $\mathbf{x}_\Sigma(t,\mathbf{x}_0)$ the state reached by Σ at time t starting from initial condition $\mathbf{x}_0$ with input sequence $\mathbf{u}_k$, $k=0,\dots,t-1$. Assume that the trajectory $\mathbf{x}_\Sigma(t,\mathbf{x}_0)$ is feasible,* i.e., *$\mathbf{x}_\Sigma(t,\mathbf{x}_0)\in\mathbb{X}$ for all t. Then, NPMHE is asymptotically convergent if $\|\hat{\mathbf{x}}_{t/t}-\mathbf{x}_\Sigma(t,\mathbf{x}_0)\|\xrightarrow{t\to\infty}0$.*

Moreover, in order to state the main convergence result, some further definitions are required.

Definition 6 *Let $\eta\geq 0$ and r be a real and an integer number respectively. Then, $\lambda(\eta,r)$ is a function defined as*

$$\lambda(\eta,r)=\frac{1-\eta^r}{1-\eta},\ \textit{if}\ \eta\neq 1,\ \textit{and}\ \lambda(\eta,r)=r,\ \textit{if}\ \eta=1$$

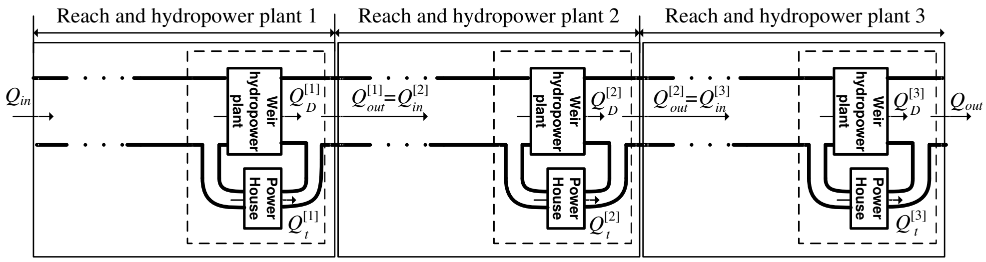

Figure 3.1: River scheme.

Definition 7 *The $\mathcal{K}_\infty$-functions $\tilde{\gamma}_{ij}$ are defined as*

$$\tilde{\gamma}_{ij}(\eta) = \max_{k\in[t-N+1:t]} \left(l_{ii}^{k-(t-N)} \gamma_{ij}(\eta) + l_{ij}\lambda(l_{ii}, k-(t-N))\eta \right) \tag{3.28}$$

Definition 8 *Given a vector $\boldsymbol{\delta} \in \mathbb{R}^M$, with components $\delta^{[i]} \geq 0$, $i = 1,\ldots,M$, define the gain map*

$$\tilde{\Gamma}(\boldsymbol{\delta}) = \begin{bmatrix} \sum_{j\in\mathscr{V}_1} \tilde{\gamma}_{1j}(\delta^{[j]}) \\ \vdots \\ \sum_{j\in\mathscr{V}_M} \tilde{\gamma}_{Mj}(\delta^{[j]}) \end{bmatrix} \tag{3.29}$$

and the diagonal operator $D : \mathbb{R}^M \to \mathbb{R}^M$ such that

$$D(\boldsymbol{\delta}) := \begin{bmatrix} (\mathrm{Id} + d_1)(\delta^{[1]}) \\ \vdots \\ (\mathrm{Id} + d_M)(\delta^{[M]}) \end{bmatrix}$$

where Id *is the identity function and $d_i \in \mathcal{K}_\infty$, $i = 1,\ldots,M$.*

Finally we need to introduce a *small gain condition*, which will be fundamental to guarantee convergence of NPMHE and is derived from [10]. Specifically, we will require that, for all $\boldsymbol{\delta} \neq 0$, with components $\delta^{[i]} \geq 0$, $i = 1,\ldots,M$ one has

$$\tilde{\Gamma} \circ D \not\geq \mathrm{Id} \tag{3.30}$$

In a few words, inequality (3.30) requires that, for all $\boldsymbol{\delta}$, there is at least one component of vector $\tilde{\Gamma}(D(\boldsymbol{\delta}))$ which strictly decreases.

Theorem 3 *If Assumptions 5, 6, 7 and 8 hold and if, for all $\boldsymbol{\delta} \neq 0$, with components $\delta^{[i]} \geq 0$, $i = 1,\ldots,M$,* (3.30) *is verified, then the NPMHE scheme is asymptotically convergent.*

Recall that a Directed Acyclic Graph (DAG) is a directed graph with no cycles. Namely, $\mathscr{G}$ is a DAG if, for all subsystems i and j, when there is a path from i to j, then there does not exist a path from j to i. After a suitable permutation of the node indexes, the adjacency matrix of a DAG is triangular.

Corollary 2 *If Assumptions 5, 6, 7 and 8 hold and if the interconnection graph is a DAG, then the NPMHE scheme is asymptotically convergent.*

3.5 An application: three cascade river reaches

In this section we apply the NPMHE algorithm to a system composed by three river reaches, which are part of a larger system describing a Hydro Power Valley (HPV), see [12]. The development of distributed predictive control techniques for the HPV requires proper estimation of the states of the reaches, i.e. levels and flow rates at different points, as well as of the disturbances, which correspond to unknown input/output terms (e.g., creeks, rain, leakages).

3.5.1 Model of the reaches

Each reach is endowed with a power house placed in a lateral channel where a turbine generates the electric power, and with a weir along the main natural river course, see Fig. 3.1. The model of a single reach is based on the classical de Saint Venant equations, i.e. mass and momentum equations, see e.g. [44], [21], [26], [27]. Letting $x \in \mathbb{R}$ be the main spatial coordinate defined by the flow direction and τ be the continuous time, in order to simplify the model, the assumptions of constant river width $W(x)$ and rectangular cross section $S(x,\tau)$ are made. Furthermore, we assume that the river friction slope is given by the Manning-Strickler equation [26].
According to the previous assumptions and denoting with $H(x,\tau)$ and $Q(x,\tau)$ the river height and the flow rate, respectively, the de Saint Venant dynamic equations can be written as

$$\begin{aligned} \frac{\partial H}{\partial \tau} &= -\frac{1}{W}\frac{\partial Q}{\partial x} \\ \frac{\partial Q}{\partial \tau} &= \frac{-2Q}{WH}\frac{\partial Q}{\partial x} + \left(\left(\frac{Q}{H}\right)^2 \frac{1}{W} - gWH\right)\frac{\partial H}{\partial x} + gWI_0H \\ &\quad - \frac{gWH}{k_{str}^2}\left(\frac{W+2H}{WH}\right)^{4/3}\left(\frac{Q}{WH}\right)^2 \end{aligned} \tag{3.31}$$

where the dependence of the variables Q and H upon x and τ has been omitted for simplicity, g is the gravitational acceleration, I_0 is the bed slope and $k_{str}(x)$ is the Strickler coefficient.

For simulation, control and estimation purposes, the model has been discretized into N_c sections along the flow direction, each one with length $\Delta x = X/N_c$, where X is the total length of the reach. To avoid unnecessary stiffness, the crossing sections of the different variables are overlapped. The flow rates Q are computed at the crossing of each section while the heights H are computed in the middle of the section, see Fig. 3.2 and the references [44], [21]. The discretization is made by the finite difference method by approximating the derivatives with the first term in the Taylor series expansions of Q and H around any point χ_i:

$$\begin{aligned} \frac{\partial Q_{2i}}{\partial \chi_{2i}} &= \frac{Q(\chi_{2i+2}) - Q(\chi_{2i})}{\chi_{2i+2} - \chi_{2i}} \\ \frac{\partial H_{2i-1}}{\partial \chi_{2i-1}} &= \frac{H(\chi_{2i+1}) - H(\chi_{2i-1})}{\chi_{2i+1} - \chi_{2i-1}} \end{aligned}$$

The boundary conditions are given by the inlet flow rate at the first reach Q_{in} and by the output flow rate Q_{out}, which is a function of the level at the end of the third reach, i.e. $Q_{out} = Q_{out}(H_{2N_c+1})$. For the second and third reaches, the inlet flow rates coincide with the outlet flow rate of the upstream reach.

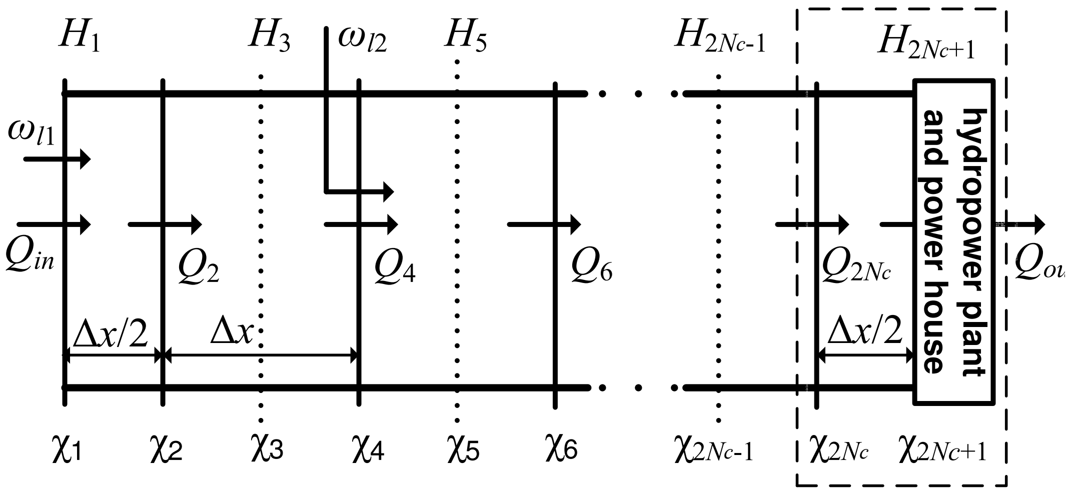

Figure 3.2: Spatial discretization of each reach. The portion of the reach enclosed in the dashed frame corresponds to the portion of each reach enclosed in the dashed frame in Fig. 3.1. ω_{l1}, ω_{l2} are flow disturbances and $\omega_{l2} = 0$ for reaches 1 and 3.

In view of these assumptions, the ODE model of each reach is

$$\begin{aligned}
\frac{dH_1}{d\tau} &= -\frac{1}{W}\frac{Q_2 - Q_{in}}{\Delta x/2} \\
\frac{dQ_2}{d\tau} &= -\frac{2Q_2}{WH_2}\frac{Q_2 - Q_{in}}{\Delta x/2} + \left(\frac{Q_2^2}{WH_2^2} - gWH_2\right)\frac{H_3 - H_1}{\Delta x} \\
&\quad -\frac{gWH_2}{k_{str}^2}\left(\frac{W+2H_2}{H_2}\right)^{4/3}\left(\frac{Q_2}{WH_2}\right)^2 + gWI_0H_2 \\
\frac{dH_{2j-1}}{d\tau} &= -\frac{1}{W}\frac{Q_{2j} - Q_{2j-2}}{\Delta x} \\
\frac{dQ_{2j}}{d\tau} &= -\frac{2Q_{2j}}{WH_{2j}}\frac{Q_{2j} - Q_{2j-2}}{\Delta x} + \left(\frac{Q_{2j}^2}{WH_{2j}^2} - gWH_{2j}\right)\frac{H_{2j+1} - H_{2j-1}}{\Delta x} \\
&\quad -\frac{gWH_{2j}}{k_{str}^2}\left(\frac{W+2H_{2j}}{WH_{2j}}\right)^{4/3}\left(\frac{Q_{2j}}{WH_{2j}}\right)^2 + gWI_0H_{2j}, \\
&\quad j = 2, \ldots, N_c \\
\frac{dH_{2N_c+1}}{d\tau} &= -\frac{1}{W}\frac{Q_{2N_c} - Q_{out}}{\Delta x/2}
\end{aligned} \tag{3.32}$$

where the dependence on τ has been omitted and the heights H_{2i}, $i = 1, \ldots, N_c$, are computed as a linear combination of the adjacent heights, i.e.

$$H_{2i} = \frac{H_{2i+1} + H_{2i-1}}{2}, \qquad i = 1, \ldots, N_c$$

The output flow rate of each reach is defined as

$$Q_{out} = Q_t + Q_D(H_{2N_c+1}) \tag{3.33}$$

where

$$Q_D(H_{2N_c+1}) = k_{weir}A_{weir}\sqrt{2g(H_{2N_c+1})} \tag{3.34}$$

k_{weir} is a parameter that depends of the characteristic of the dam, A_{weir} is the weir cross-sectional area, Q_D is the flow rate through the weir of the dam and Q_t is the flow rate through the channel and the power house, assumed constant in the considered state estimation problem.
In the following each reach has been divided into $N_c = 5$ cells.

Length of each reach	$4000[m]$
Width of each reach W	$100[m]$
Strickler coefficient k_{str}	$30[m^{1/3}/s]$
Slope of the bed of each reach I_0	$0.0033[-]$
Section of the weir of the dam A_{weir}	$18.26[m^2]$
Discharge coefficient, weir of the dam, k_{weir}	$0.6[-]$
Nominal flow rate through the turbine Q_t	$100[m^3/s]$

Table 3.1: Reach Data

3.5.2 Disturbances model

In order to test the capability of the proposed MHE scheme to estimate unknown disturbances, it is assumed that each reach is affected by unmeasurable inlet flows. Specifically, as shown in Fig. 3.2, a flow rate ω_{l1} is forced at the beginning of each reach and represents the variation of the inflow Q_{in} due to the variations of the concession level of an upstream dam. Moreover, an additional flow rate ω_{l2} is placed at the input of the third cell of the second reach to model the presence of an unknown affluent.

Both ω_{l1} and ω_{l2} are generated as the sum of a constant term ($\overline{\omega}_1$ and $\overline{\omega}_2$) and the state (d_1 and d_2) of a first-order stable system fed by zero-mean White Gaussian Noises (WGNs) $w_1(k)$ and $w_2(k)$ with variances σ_1^2 and σ_2^2, respectively. Saturation constraints are included to impose that these disturbances are non negative.

3.5.3 River data and available measurements

The three reaches have the same geometric characteristics, summarized in Table 3.1. In nominal stationary conditions, the considered constant flow rate is $\overline{Q} = 300[m^3/s]$, while the values of the height are $\overline{H}_1 = 3.83[m]$, $\overline{H}_3 = 7.11[m]$, $\overline{H}_5 = 10.4[m]$, $\overline{H}_7 = 13.7[m]$, $\overline{H}_9 = 17[m]$. As for the disturbances, the following values have been used: $\overline{\omega}_1 = 10[m^3/s], \overline{\omega}_2 = 30[m^3/s], \sigma_1^2 = \sigma_2^2 = 5$, while the filters have gain equal to 0.5 and time constant $10^5[s]$. Moreover, it is assumed that the inflow Q_{in} of the first reach is known, as well as the flow rates through the turbines, while for estimation and control purposes three measurements are available for any reach, namely the levels H_1 and H_5 and the flow rate Q_8 in the first and third reaches, and the variables H_1, H_7 and Q_4 in the second reach. These measurements are also affected by noise; specifically, a WGN with zero mean and variance $\sigma^2 = 0.1$ is added to the level measures, while a zero mean WGN with unitary variance corrupts the flow measure. Remarkably, the measured flow Q_8 does not correspond to the inflow of the downstream reach, see eqs. (3.33) and (3.34), so that there is an effective coupling between the estimation problems. This motivates the use of the NPMHE scheme presented in the previous sections.

3.5.4 MHE and simulation results

The NPMHE algorithm described in Sections 2-4 is applied to the three reaches viewed as a system with a cascade structure; therefore, Corollary 2 is automatically verified. The reaches are described by the equations (3.32), while the mutual influences are due to the relations (3.33) and (3.34). As such, the state of the subsystems are $x^{[i]} = (H_1^{[i]}, Q_2^{[i]}, H_3^{[i]}, Q_4^{[i]}, H_5^{[i]}, Q_6^{[i]}, H_7^{[i]}, Q_8^{[i]}, H_9^{[i]})$, $i = 1,2,3$, while the inputs are $u^{[i]} = Q_t, \quad i = 1,2,3$. Positivity constraints on all the estimated states have been imposed and flows $Q_j^{[i]}$ $(i = 1,2,3,\ j = 2,4,6,8)$ are constrained to be smaller than $450[m^3/s]$.
The models of the reaches have then been discretized with a sampling time $\Delta\tau = 60[s]$ to implement

the distributed MHE algorithm, that assumes discrete-time systems. In the discrete-time model so obtained, it has been assumed that the state disturbance $\mathbf{w}_t$ (see (3.1)) acts on the states H_1, Q_2 for the first and third reaches, and on H_1, Q_2, H_5, Q_6 for the second reach.
The stage cost and the initial penalty (see (3.12)) are given by equations (3.25) and (3.26). where $\Pi^{[i]}_{t-N/t-1} = 10 I_9$, $i = 1,2,3$. The matrices $\mathscr{Q}^{[i]}$ are diagonal with elements equal to $3.33 \cdot 10^4$ corresponding to the non-zero components of the disturbance $\mathbf{w}_t$, and equal to very small values corresponding to the zero components. Also the matrices $\mathscr{R}^{[i]}$ have been chosen as diagonal, with elements equal to 200 for the level estimation errors and to $2 \cdot 10^3$ for the flow rate estimation errors.
The simulation experiments have been performed with MATLAB and optimizations are carried out with the TOMLAB optimization environment [23]. We add a sinusoidal variation of amplitude $\pm 30 [m^3/s]$ and period of about $2.3[h]$ to the nominal inlet flow rate Q_{in}. With reference to the first reach, Fig. 3.3 shows the true and estimated values of the flow rate Q_2 and of the levels H_1 and H_9 at the beginning and at the end of the reach. Fig. 3.4 depicts the true and estimated disturbance acting on the initial section of the reach. These results clearly show that, after an initial negligible transient mainly due to the optimization procedure, all the estimates converge to the true values.

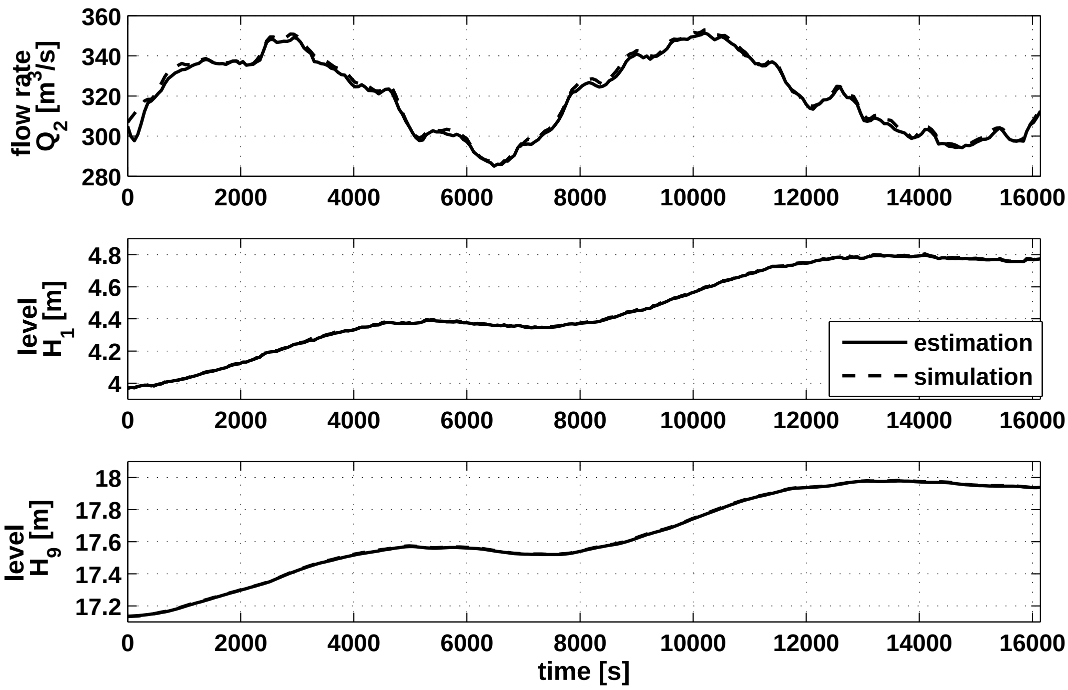

Figure 3.3: Levels H_1, H_9 and flow rate Q_2 of the first reach.

Fig. 3.5 illustrates the results for the second reach, in particular the values and estimates of Q_2, H_1 and of Q_6, which is the flow rate of the reach after the additional inlet flow described as a disturbance term (see Fig.3.2). The values of the true and estimated disturbances acting on the second reach are shown in Figs.3.6 and 3.7. It is apparent that the proposed scheme is able to correctly determine the state and disturbance estimates even though the additional input flow rate is not directly measured, but computed on the basis of the estimates performed for the first reach.

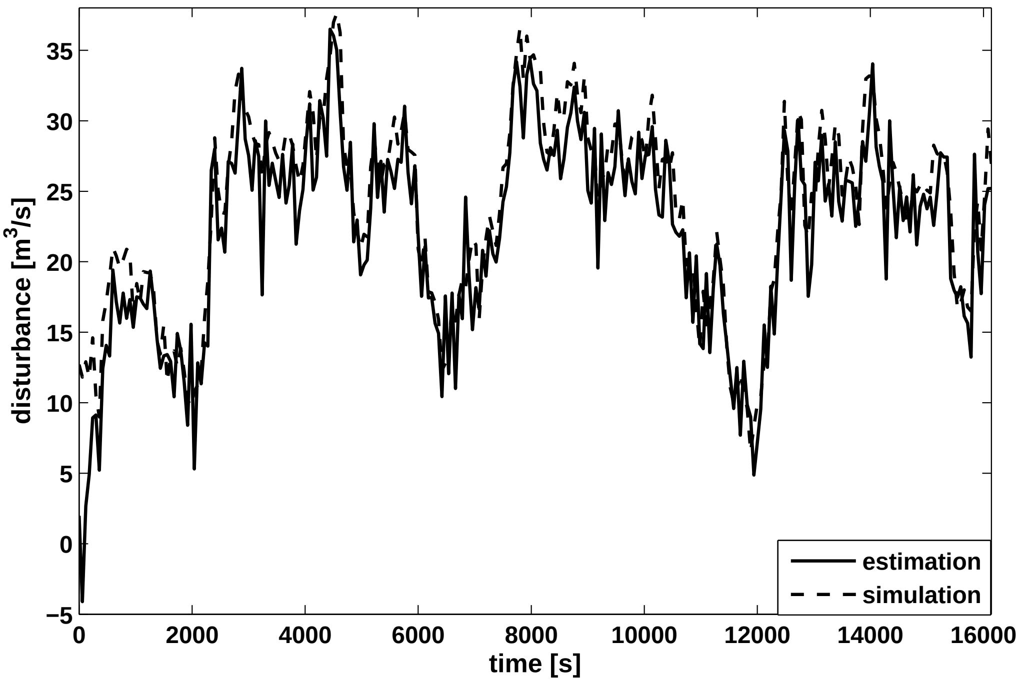

Figure 3.4: Disturbance at the beginning of the first reach.

Finally, in Fig. 3.8 and in Fig. 3.9 the estimates of Q_2, H_1, H_9 and of the disturbance acting at the initial section of the third reach are depicted, showing estimation performances comparable with those obtained on the previous reaches.

3.6 Proofs

Since system (3.5) is time-invariant, for $N \geq \bar{n}^o = \max\{n_i^o\}$, Assumption 6 guarantees that

$$\begin{array}{c} \|x_{t-N}^{[i]} - x_{t-N}^{*[i]}\| \leq \gamma_w^{[i]}(\|w_k^{[i]} - w_k^{*[i]}\|_{[t-N:t-1]}) \\ +\gamma_y^{[i]}(\|y_k^{[i]} - y_k^{*[i]}\|_{[t-N:t]}) + \sum_{j\in\mathscr{N}_i}\gamma_{ij}(\|x_k^{[j]} - x_k^{*[j]}\|_{[t-N:t-1]}) \end{array} \tag{3.35}$$

where $y_k^{[i]}$ and $y_k^{*[i]}$ are the output sequences stemming from $(w_k^{[i]}, u_k^{[i],x}, u_k^{[i]})$ and $(w_k^{*[i]}, u_k^{*[i],x}, u_k^{[i]})$ with initial conditions $x_{t-N}^{[i]}$ and $x_{t-N}^{*[i]}$, respectively.

The first step towards the convergence of the NPMHE scheme is the following lemma.

Lemma 4 *If Assumptions 7 and 8 hold then*

$$\sum_{k=t-N}^{t} \mathbf{L}(\hat{\mathbf{w}}_{k/t}, \hat{\mathbf{v}}_{k/t}) \xrightarrow{t\to\infty} 0 \tag{3.36}$$

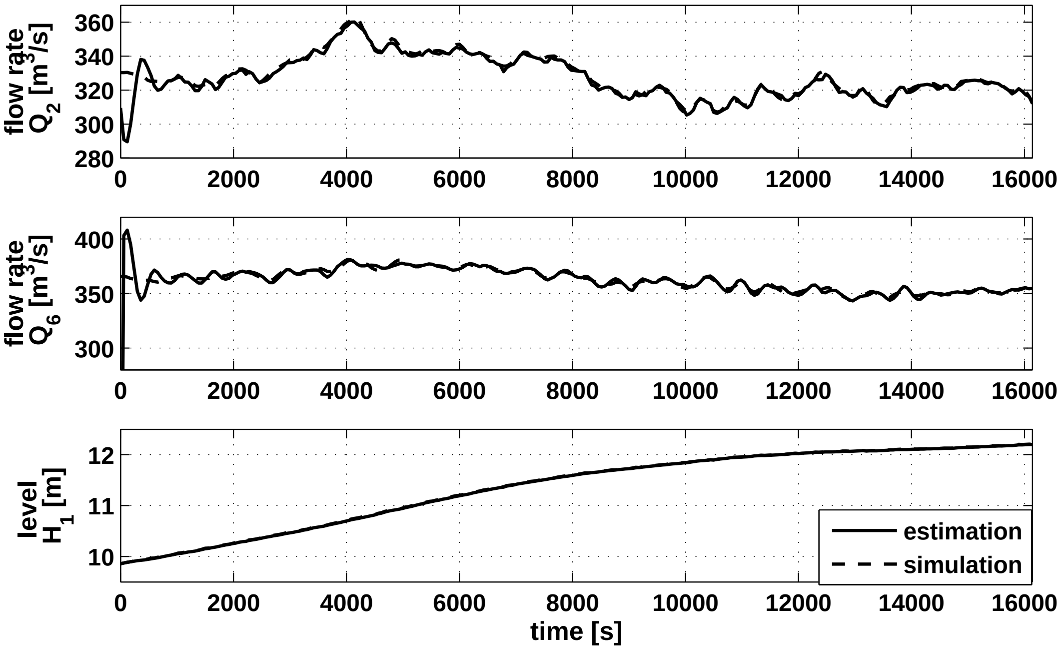

Figure 3.5: Level H_1 flow rates Q_2 and Q_6 of the second reach.

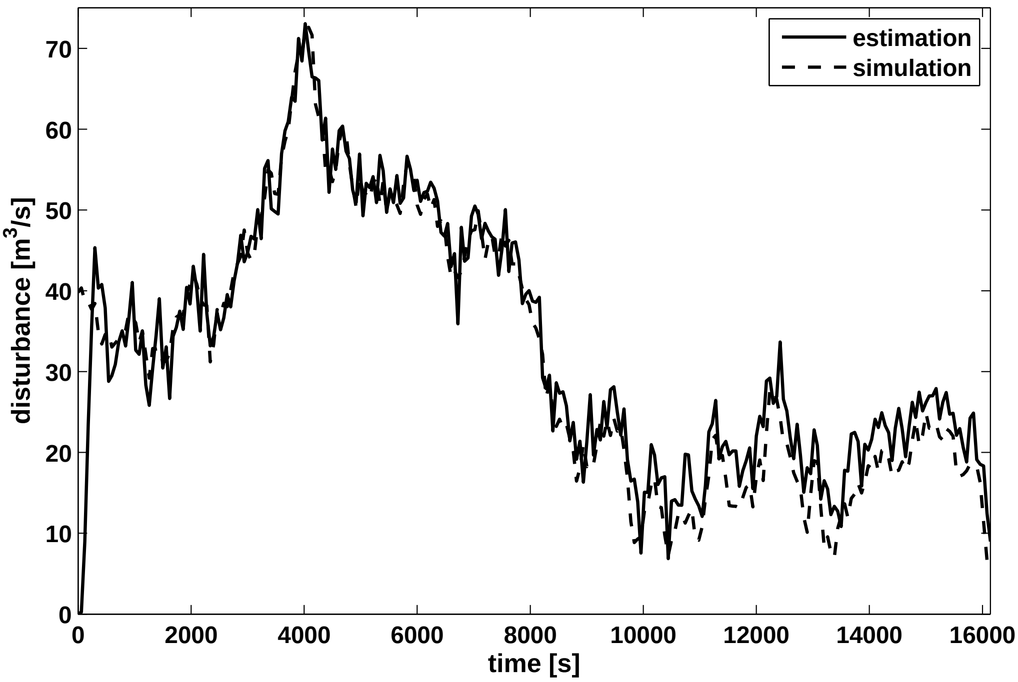

Figure 3.6: Disturbance at the beginning of the second reach.

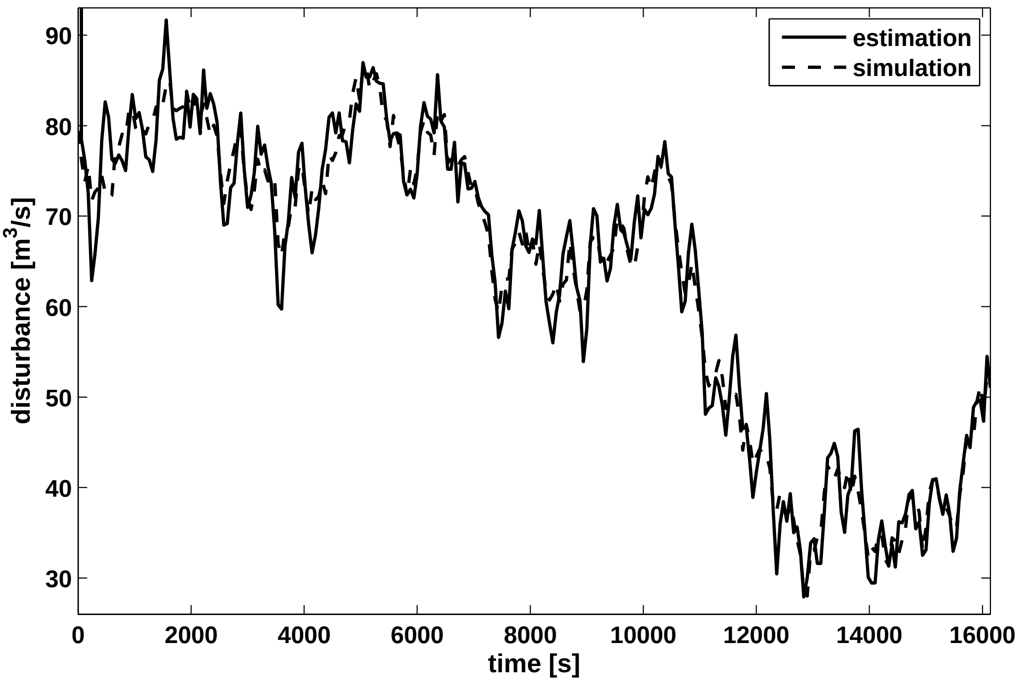

Figure 3.7: Disturbance in the middle of the second reach.

Proof of Lemma 4 The proof is similar to the one in Proposition 5 in [40]. For all $t \geq 0$, in view of (3.23) one has

$$\mathbf{\Theta}_t^* - \mathbf{\Theta}_{t-1}^* \geq \sum_{k=t-N}^{t} \mathbf{L}(\hat{\mathbf{w}}_{k/t}, \hat{\mathbf{v}}_{k/t}) \tag{3.37}$$

Note that, by optimality $\mathbf{\Theta}_t^* \leq \mathbf{\Xi}_{[t-N+1:t]/t}(\{\mathbf{x}_\Sigma(k,\mathbf{x}_0)\}_{k=t-N+1}^t)$ $\forall t > N$. The trajectory stemming from $\mathbf{x}_\Sigma(t-N,\mathbf{x}_0)$, $\hat{\mathbf{w}}_k^* = \tilde{\mathbf{f}}(\mathbf{x}_\Sigma(k,\mathbf{x}_0),\mathbf{x}_\Sigma(k,\mathbf{x}_0),\mathbf{u}_k) - \tilde{\mathbf{f}}(\mathbf{x}_\Sigma(k,\mathbf{x}_0),\hat{\mathbf{x}}_{k/t-1},\mathbf{u}_k)$ for $k = t-N, \ldots, t-1$ and $\hat{\mathbf{w}}_t^* = 0$ is $\mathbf{x}_\Sigma(k,\mathbf{x}_0)$ for $k = t-N+1,\ldots,t$, and hence it is feasible. Since $\mathbf{y}_{[t-N:t]}$ corresponds to the deterministic system output (see Definition 5), it follows that $\hat{\mathbf{v}}_k = \mathbf{y}_k - \mathbf{h}(\mathbf{x}_\Sigma(k,\mathbf{x}_0),\mathbf{u}_t) = 0$ for all $k = t-N,\ldots,t$. Moreover, by optimality

$$\begin{aligned}&\mathbf{\Xi}_{[t-N+1:t]/t}(\{\mathbf{x}_\Sigma(k,\mathbf{x}_0)\}_{k=t-N+1}^t)\\ &\qquad \leq \mathbf{J}(t-N,t,\mathbf{x}_\Sigma(t-N,\mathbf{x}_0),\hat{\mathbf{w}}^*,0,\mathbf{\Gamma}_{t-N})\end{aligned}$$

From (3.15), one has

$$\begin{aligned}&\mathbf{J}(t-N,t,\mathbf{x}_\Sigma(t-N,\mathbf{x}_0),\hat{\mathbf{w}}^*,0,\mathbf{\Gamma}_{t-N}) =\\ &= \sum_{k=t-N}^{t-1} \mathbf{L}(\hat{\mathbf{w}}_k^*,0) + \mathbf{\Gamma}_{t-N}(\mathbf{x}_\Sigma(t-N,\mathbf{x}_0);\hat{\mathbf{x}}_{t-N/t-1}),\end{aligned}$$

and in view of (3.24), $\mathbf{\Theta}_t^* \leq \mathbf{\Xi}_{[t-N:t-1]/t-1}(\{\mathbf{x}_\Sigma(k,\mathbf{x}_0)\}_{k=t-N}^{t-1})$. We can iterate this procedure and prove that

$$\mathbf{\Theta}_t^* \leq \Gamma_0^*(\mathbf{x}_0;\mathbf{m}_0) \leq \boldsymbol{\gamma}_0(\|\mathbf{x}_0 - \mathbf{m}_0\|) \tag{3.38}$$

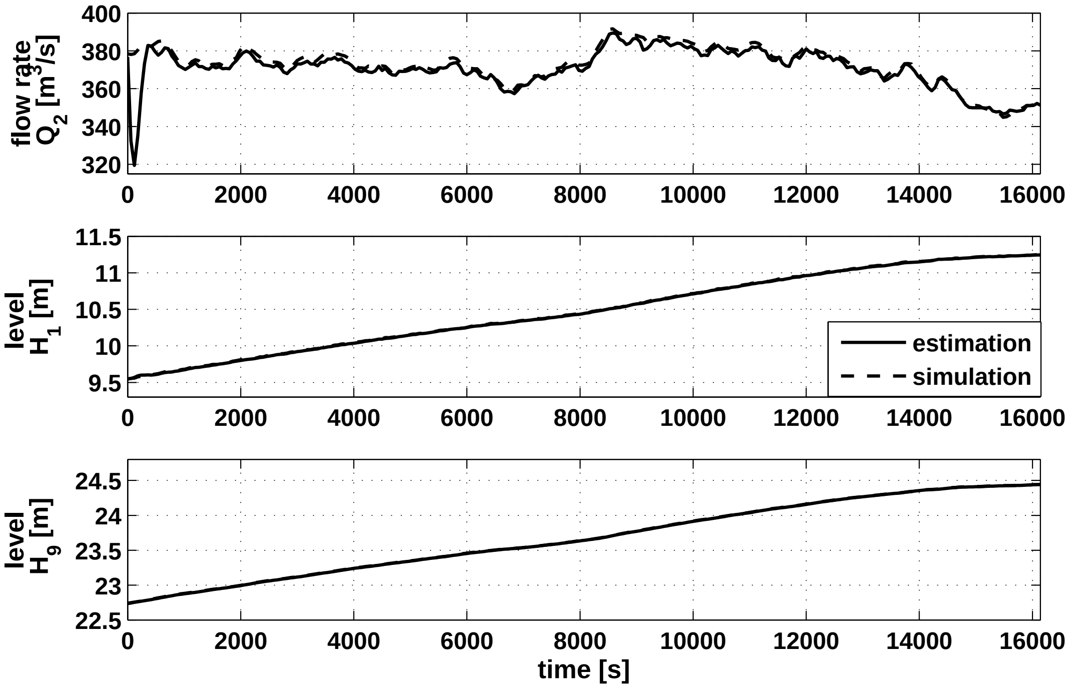

Figure 3.8: Levels H_1, H_9 and flow rate Q_2 of the third reach.

for all t, for any $\mathbf{x}_0 \in \mathbb{X}$, where $\mathbf{m}_0 \in \mathbb{X}$ is the prior estimate of $\mathbf{x}_0$ and where γ_0^* is a suitable $\mathscr{K}_\infty$ function, in view of (3.22b), which derives from Assumption 7.
Finally, from (3.37) the sequence $\mathbf{\Theta}_t^*$ is increasing and from (3.38) it is bounded. Therefore, the sequence $\mathbf{\Theta}_t^*$ converges and, from (3.37), equation (3.36) follows. ■

In view of Lemma 4, the proof of Theorem 3 can be devised.

Proof of Theorem 3 Since Assumptions 7 and 8 hold, by Lemma 4 equation (3.36) is guaranteed. In view of (3.22a), it implies that

$$\max_{k\in[t-N:t]}(\|\hat{\mathbf{v}}_{k/t}\|,\|\hat{\mathbf{w}}_{k/t}\|) \xrightarrow{t\to\infty} 0 \tag{3.39}$$

Notice that, in the noiseless case ($w_\Sigma^{[i]}(k)=0$ for all k and $i=1,\ldots,M$), for any t, the trajectory $\mathbf{x}_\Sigma(t,\mathbf{x}_0)$ is generated by the system

$$\mathbf{x}_\Sigma(t+1,\mathbf{x}_0)=\tilde{\mathbf{f}}(\mathbf{x}_\Sigma(t+1,\mathbf{x}_0),\mathbf{x}_\Sigma(t+1,\mathbf{x}_0),\mathbf{u}_t) \tag{3.40}$$

and the output signal, for each sub-system, is

$$y_k^{[i]}=h^{[i]}(x_\Sigma^{[i]}(k,\mathbf{x}_0),u_k^{[i]})$$

and $\hat{v}_{k/t}^{[i]}=y_k^{[i]}-h^{[i]}(\hat{x}_{k/t}^{[i]},u_k^{[i]})$. Recalling (3.35), we obtain that

$$\begin{aligned}\|\hat{x}_{t-N/t}^{[i]}-x_\Sigma^{[i]}(t-N,\mathbf{x}_0)\| &\le \gamma_w^{[i]}(\|w_\Sigma^{[i]}(k)-\hat{w}_k^{[i]}\|_{[t-N:t-1]})\\ &+\gamma_y^{[i]}(\|\hat{v}_{k/t}^{[i]}\|_{[t-N:t]})+\sum_{j\in\mathscr{N}_i}\gamma_{ij}(\|\hat{x}_{k/t-1}^{[j]}-x_\Sigma^{[j]}(k,\mathbf{x}_0)\|_{[t-N:t-1]})\end{aligned} \tag{3.41}$$

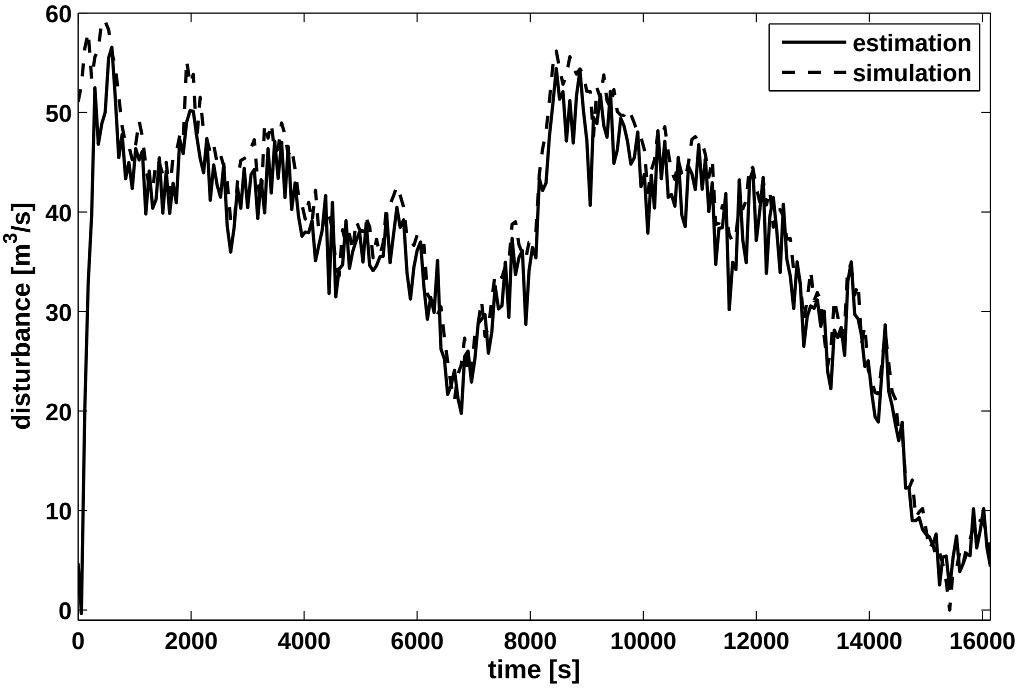

Figure 3.9: Disturbance at the beginning of the third reach.

From (3.39) we obtain that, for all $i = 1, \dots, M$ there exists a positive sequence $\alpha_t^{[i]}$ satisfying $\alpha_t^{[i]} \xrightarrow{t\to\infty} 0$ such that

$$\begin{aligned} &\|\hat{x}^{[i]}_{t-N/t} - x^{[i]}_{\Sigma}(t-N,\mathbf{x}_0)\| \leq \\ &\leq \textstyle\sum_{j\in\mathcal{N}_i} \gamma_{ij}(\|\hat{x}^{[j]}_{k/t-1} - x^{[j]}_{\Sigma}(k,\mathbf{x}_0)\|_{[t-N:t-1]}) + \alpha_t^{[i]} \end{aligned} \tag{3.42}$$

Recall that $u^{[i],x}_{k/t-1} = \{\hat{x}^{[j]}_{k/t-1}, j \in \mathcal{N}_i\}$ and, in the noiseless setting, $u^{[i],x}_{k} = \{x^{[j]}_{\Sigma}(k,\mathbf{x}_0), j \in \mathcal{N}_i\}$. For $k \geq t-N$, in view of (3.9b)

$$\hat{x}^{[i]}_{k+1/t} = f^{[i]}(\hat{x}^{[i]}_{k/t}, u^{[i],x}_{k/t-1}, u^{[i]}_k) + \hat{w}^{[i]}_{k/t} \tag{3.43}$$

while $x^{[i]}_{\Sigma}(k,\mathbf{x}_0)$ stems from (3.40), that is

$$x^{[i]}_{\Sigma}(k+1,\mathbf{x}_0) = f^{[i]}(x^{[i]}_{\Sigma}(k,\mathbf{x}_0), u^{[i],x}_k, u^{[i]}_k) \tag{3.44}$$

Defining $\Delta^{[i]}_{k_1/k_2} = \hat{x}^{[i]}_{k_1/k_2} - x^{[i]}_{\Sigma}(k_1,\mathbf{x}_0)$, we obtain, from (3.43) and (3.44)

$$\begin{aligned} \Delta^{[i]}_{k+1/t} =& f^{[i]}(x^{[i]}_{\Sigma}(k,\mathbf{x}_0) + \Delta^{[i]}_{k/t}, \{x^{[j]}_{\Sigma}(k,\mathbf{x}_0) + \Delta^{[j]}_{k/t-1}\}_{j\in\mathcal{N}_i}, u^{[i]}_k) + \\ & - f^{[i]}(x^{[i]}_{\Sigma}(k,\mathbf{x}_0), \{x^{[j]}_{\Sigma}(k,\mathbf{x}_0)\}_{j\in\mathcal{N}_i}, u^{[i]}_k) + \hat{w}^{[i]}_{k/t} \end{aligned} \tag{3.45}$$

From (3.45) and (3.6) (which derives from Assumption 5) it follows that:

$$\begin{aligned}\|\Delta^{[i]}_{k+1/t}\| \le & \| f^{[i]}(x^{[i]}_{\Sigma}(k,\mathbf{x}_0)+\Delta^{[i]}_{k/t},\{x^{[j]}_{\Sigma}(k,\mathbf{x}_0)+\Delta^{[j]}_{k/t-1}\}_{j\in\mathcal{N}_i},u^{[i]}_k) \\ & - f^{[i]}(x^{[i]}_{\Sigma}(k,\mathbf{x}_0),\{x^{[j]}_{\Sigma}(k,\mathbf{x}_0)\}_{j\in\mathcal{N}_i},u^{[i]}_k)\| + \|\hat{w}^{[i]}_{k/t}\| \\ \le & l_{ii}\|\Delta^{[i]}_{k/t}\| + \sum_{j\in\mathcal{V}_i} l_{ij}\|\Delta^{[j]}_{k/t-1}\| + \|\hat{w}^{[i]}_{k/t}\| \end{aligned} \tag{3.46}$$

Iterating equation (3.46) we obtain that, for $k=t-N+1,\dots,t$

$$\begin{aligned}\|\Delta^{[i]}_{k/t}\| \le & l_{ii}^{k-(t-N)}\|\Delta^{[i]}_{t-N/t}\| + \sum_{j\in\mathcal{V}_i} l_{ij}\Big(\sum_{r=0}^{k-(t-N)-1} l^r_{ii}\|\Delta^{[j]}_{k-1-r/t-1}\|\Big) \\ & + \sum_{r=0}^{k-(t-N)-1} l^r_{ii}\|\hat{w}^{[i]}_{k-1-r/t}\| \end{aligned} \tag{3.47}$$

Defining $\delta^{[i]}_t = \max_{k\in[t-N+1:t]}\|\Delta^{[i]}_{k/t}\|$ and $\alpha^{w,[i]}_t = \max_{k\in[t-N:t-1]}\|\hat{w}^{[i]}_{k/t}\|$ we can write, from (3.47)

$$\begin{aligned}\|\Delta^{[i]}_{k/t}\| \le & l_{ii}^{k-(t-N)}\|\Delta^{[i]}_{t-N/t}\| + \sum_{j\in\mathcal{N}_i} l_{ij}\lambda(l_{ii},k-(t-N))\delta^{[j]}_{t-1} \\ & + \lambda(l_{ii},k-(t-N))\alpha^{w,[i]}_t \end{aligned} \tag{3.48}$$

where $\lambda(\cdot,\cdot)$ is given in Definition 6. In view of (3.42), $\|\Delta^{[i]}_{t-N/t}\| \le \sum_{j\in\mathcal{N}_i}\gamma_{ij}(\delta^{[j]}_{t-1})+\alpha^{[i]}_t$, from which it follows that

$$\begin{aligned}\|\Delta^{[i]}_{k/t}\| \le & \sum_{j\in\mathcal{N}_i}\Big(l_{ii}^{k-(t-N)}\gamma_{ij}(\delta^{[j]}_{t-1}) + l_{ij}\lambda(l_{ii},k-(t-N))\delta^{[j]}_{t-1}\Big) \\ & + \lambda(l_{ii},k-(t-N))\alpha^{w,[i]}_t + l_{ii}^{k-(t-N)}\alpha^{[i]}_t \end{aligned} \tag{3.49}$$

Therefore one concludes that

$$\begin{aligned}\delta^{[i]}_t \le & \sum_{j\in\mathcal{N}_i}\max_{k\in[t-N+1:t]}(l_{ii}^{k-(t-N)}\gamma_{ij}(\delta^{[j]}_{t-1}) \\ & + l_{ij}\lambda(l_{ii},k-(t-N))\delta^{[j]}_{t-1}) \\ & + \max_{k\in[t-N+1:t]}\Big(\lambda(l_{ii},k-(t-N))\alpha^{w,[i]}_t + l_{ii}^{k-(t-N)}\alpha^{[i]}_t\Big) \end{aligned} \tag{3.50}$$

We define

$$\tilde{\alpha}^{[i]}_t = \max_{k\in[t-N+1:t]}\Big(\lambda(l_{ii},k-(t-N))\alpha^{w,[i]}_t + l_{ii}^{k-(t-N)}\alpha^{[i]}_t\Big)$$

which, in view of (3.39), is an asymptotically vanishing term. Furthermore, in view of Definition 7, we can write (3.50) as

$$\delta^{[i]}_t \le \sum_{j\in\mathcal{V}_i}\tilde{\gamma}_{ij}(\delta^{[j]}_{t-1}) + \tilde{\alpha}^{[i]}_t \tag{3.51}$$

The stability of the system of interconnected equations (3.51) can be analyzed by means of the small gain condition given in [10].

Define a vector $\boldsymbol{\delta}\in\mathbb{R}^M$, with components $\delta^{[i]}\ge 0$, $i=1,\dots,M$. Since, by definition, for all i, the i-th component of $\tilde{\Gamma}(\boldsymbol{\delta})$ does not depend on $\delta^{[i]}$ and is a $\mathscr{K}_\infty$ function of $\delta^{[j]}$, $j \in \mathscr{N}_i$, it is easy to see that the system of equations (3.51) is asymptotically stable (i.e., $\delta_t^{[i]} \to 0$ as $t \to \infty$ for all $i = 1, \dots, M$) if the map $\tilde{\Gamma}$ satisfies the small gain condition (3.30). ■

Proof of Corollary 2

Recall that $\mathscr{L}$ is defined in Section 3.2 as the matrix collecting the Lipschitz constants l_{ij}, defined in (3.6). If the system partition induces a DAG and since Assumption 5 holds, $\mathscr{L}$ is (lower) triangular, after a suitable permutation of the subsystem indexes. Therefore, without loss of generality, we have $l_{ij} = 0$ if $j > i$. Also, since Assumption 6 holds, similar arguments apply to the $\mathscr{K}$-functions γ_{ij} in (3.8) i.e., for all $i = 1, \dots, M$, γ_{ij} is defined only for $j < i$. This, according to (3.28), gives rise to a map $\tilde{\Gamma}(\boldsymbol{\delta})$ whose i-th element depends only upon $\delta^{[j]}$, with $j < i$, for all $i = 1, \dots, M$.

Now we prove that, since $\tilde{\Gamma}$ has such a structure and its entries are $\mathscr{K}$-functions, the small gain condition is verified. Notice that, by definition, $\delta^{[i]} \geq 0$ for all $i = 1, \dots, M$. Note that any admissible $\boldsymbol{\delta}$ satisfies the following: there exists an index i such $\delta^{[j]} = 0$ for all $j < i$ and $\delta^{[i]} > 0$ (if $i = 1$ this corresponds only to the condition $\delta^{[1]} > 0$). In view of its structure, the i-th entry of $\tilde{\Gamma}$ is equal to zero i.e., $\tilde{\Gamma}_i = 0$. Therefore, for all $\boldsymbol{\delta}$, there exists an index i such that $\tilde{\Gamma}(D(\boldsymbol{\delta}))_i < \delta^{[i]}$. This corresponds to the small gain condition (3.30). Being Assumptions 5, 6,7, 8 verified, we resort to Theorem 3 which guarantees asymptotic convergence of the NPMHE scheme. ■

Chapter 4

Variance estimation - adaptive tuning of moving horizon estimators

4.1 Introduction

The distributed estimation algorithms based on the MHE approach developed in the HD-MPC project and partially described in the previous chapters of this deliverable require the a-priori knowledge of the covariances of the noises affecting the system states and outputs, which are generally unknown. This is a serious drawback which could prevent one from achieving satisfactory results, and particular attention must be placed to the tuning phase.
Algorithms for covariance estimation have been already proposed in the literature; among them, the most significant and promising ones have already been reviewed and tested in deliverable D5.2. Specifically, reference has been made to the so-called correlation approach developed by Mehra, see [28, 29], and to the Autocovariance Least Squares (ALS) method described in [31], which has proved to be the most effective one, since it outperforms significantly the one proposed in [28].
For all the above reasons, in this chapter a simple adaptive covariance estimation algorithm based on the ALS approach is developed and used for the on-line tuning of the weights used in MHE. Basically, starting form the estimation error computed on-line, this algorithm adaptively updates the noise variances, which are actually used as weights in the MHE performance index. The method is then applied to a couple of significant test cases with excellent results, so that it is believed that the proposed approach can be successfully used in the majority of cases.
The chapter is organized as follows. First, the problem is stated, the performance index used in MHE is recalled and some preliminary definitions are reported. Then the ALS algorithm, already extensively presented in deliverable D5.2 is briefly summarized. The adaptive method for the on-line update of the MHE tuning parameters is then given and tested in the considered simulation examples.

4.2 Problem Statement

Consider a linear, time-invariant, discrete-time model:

$$\begin{aligned} x_{t+1} &= Fx_t + Bu_t + Gw_t \\ z_t &= Hx_t + v_t \end{aligned} \tag{4.1}$$

where $x \in \mathbb{R}^n$ is the system state, $F \in \mathbb{R}^{n\times n}$ is the transition matrix, $B \in \mathbb{R}^{n\times m}$ is the control matrix, $G \in \mathbb{R}^{n\times g}$ is the disturbance matrix, $z \in \mathbb{R}^p$ is the observation vector, and $H \in \mathbb{R}^{p\times n}$ is the observation matrix. Note that $\{u_t\}_{t=0}^{N_d}$, $\{w_t\}_{t=0}^{N_d}$, and $\{v_t\}_{t=0}^{N_d}$ are the control, the state uncertainty vector (or process-noise), and the measurement noise sequences respectively, with N_d the size of the sequences. The disturbances v and w are zero-mean Gaussian white noises with R_v and Q_w as covariance matrices, respectively.

Now assume that the state estimates are computed using the linear, time-invariant state estimator:

$$\begin{aligned} \hat{x}_{t+1|t} &= F\hat{x}_{t|t} + Bu_t \\ \hat{x}_{t|t} &= \hat{x}_{t|t-1} + L\left[z_t - H\hat{x}_{t|t-1}\right] \end{aligned} \tag{4.2}$$

where L is the observer gain. From (4.1), (4.2) it follows that the dynamics of the state estimation error $\varepsilon_{t|t-1} = x_t - \hat{x}_{t|t-1}$ is given by:

$$\begin{aligned} \varepsilon_{t+1|t} &= Fx_t + Bu_t + Gw_t - F\left[\hat{x}_{t|t-1} + L\left(z_t - H\hat{x}_{t|t-1}\right)\right] - Bu_t \\ &= Fx_t + Gw_t - F(I-LH)\hat{x}_{t|t-1} - FLz_t \\ &= Fx_t + Gw_t - F(I-LH)\hat{x}_{t|t-1} - F(Hx_t + v_t) \end{aligned} \tag{4.3}$$

leading to

$$\varepsilon_{t+1|t} = F(I-LH)\varepsilon_{t|t-1} + Gw_t - FLv_t \tag{4.4}$$

Moreover, assuming that $E\{\varepsilon_{t|t-1}\} = 0$, where $E\{\cdot\}$ denotes the statistical expectation, the prediction error covariance is defined as:

$$M_t = E\left\{\varepsilon_{t|t-1}\varepsilon_{t|t-1}^T\right\} \tag{4.5}$$

A very well known and effective way to compute the gain L is to resort to the Kalman filtering approach, which requires to know the true covariance matrices Q_w and R_v. In the following, we denote as M the stationary value of $M(t)$, computed through the algebraic Riccati equation.
As already shown in deliverable D5.2, the use of wrong covariances leads to a suboptimal estimation. Define the L-innovations as follows

$$\mathscr{X}_t = z_t - H\hat{x}_{t|t-1} \tag{4.6}$$

The variance estimation problem reduces to find the true matrices Q_w and R_v using real data from the innovations (4.6), with the final goal of computing the optimal prediction error covariance matrix and the optimal filter gain.
An analogous problem is related to the use of MHE, which are based on the solution of the following minimization problem:

$$\begin{aligned} &\min_{\hat{x}_{t-N},\{\hat{w}_k\}_{t-N}^{t-1}} \sum_{k=t-N}^{t-1} \|\hat{v}_k\|^2_{R_v^{-1}} + \|\hat{w}_k\|^2_{Q_w^{-1}} + \|\hat{x}_{t-N} - \hat{x}_{t-N/[t-2N:t-N-1]}\|^2_{\Pi_{t-N}^{-1}} \\ &\text{s.t.} \quad \begin{cases} \hat{x}_{k+1} &= F\hat{x}_k + \hat{w}_k \\ \hat{v}_k &= z_k - H\hat{x}_k \end{cases} \\ &\text{s.t.} \quad x \in \mathbb{X}, \quad w \in \mathbb{W}, \quad v \in \mathbb{V}, \quad \text{and (4.1)} \end{aligned} \tag{4.7}$$

where $\mathbb{X}$, $\mathbb{W}$, and $\mathbb{V}$ are polyhedral and convex sets with $0 \in \mathbb{W}$, and $0 \in \mathbb{V}$, N is the moving horizon window, $\hat{x}_{t-N}$ and $\hat{w}_k$, $k = t-N,\ldots,t-1$ are the optimization variables which correspond to the initial condition of the state in the moving window and to the model disturbance sequence respectively, t is the current time. Finally, the term

$$\Xi_{t-N}(\hat{x}_{t-N}) = \|\hat{x}_{t-N} - \hat{x}_{t-N/[t-2N:t-N-1]}\|^2_{\Pi^{-1}_{t-N}}$$

is denoted *arrival cost* [40], where $\hat{x}_{t-N/[t-2N:t-N-1]}$ is given as the result of the optimization problem (4.7) solved at time $t-N$ with available data $z_{t-2N}, \dots, z_{t-N-1}$.

For linear systems, in the unconstrained case, and by a proper choice of the arrival cost, it has been shown the equivalence between the Kalman predictor and MHE, see [40, 42, 41]. In this scenario the tuning matrices R_v and Q_w must be chosen as the covariances of the noises v and w, respectively. Therefore, the problem of properly tuning these matrices is fundamental for obtaining suitable estimation performance for MHE methods as well.

4.2.1 Autocovariance Least Squares -ALS

The ALS method presented in [31] and already extensively described in deliverable D5.2 is now briefly summarized.
Consider the dynamic evolution of the state prediction error, $\varepsilon_t = x_t - \hat{x}_{t|t-1}$, from (4.4):

$$\varepsilon_{t+1} = \underbrace{(F - FLH)}_{\bar{F}}\varepsilon_t + \underbrace{\begin{bmatrix} G & -FL \end{bmatrix}}_{\bar{G}} \underbrace{\begin{bmatrix} w_t \\ v_t \end{bmatrix}}_{\bar{w}_t} \tag{4.8}$$

Then, the state-space model of the L-innovations is defined as:

$$\begin{aligned} \varepsilon_{t+1} &= \bar{F}\varepsilon_t + \bar{G}\bar{w}_t \\ \mathscr{Z}_t &= H\varepsilon_t + v_t \end{aligned} \tag{4.9}$$

In the sequel, the following conditions are assumed to hold:

- The pair (F, H) is detectable.
- The transition matrix of the estimation error dynamics is stable.
- $E(\varepsilon_0) = 0, \quad \mathrm{Cov}(\varepsilon_0) = M_0$

Under stationarity conditions, we have that

$$\begin{aligned} E\left\{\mathscr{Z}_t \mathscr{Z}_t^T\right\} &= HMH^T + R_v \\ E\left\{\mathscr{Z}_{t+j} \mathscr{Z}_t^T\right\} &= H\bar{F}^j MH^T - H\bar{F}^{j-1}FLR_v, \quad j \geq 1 \end{aligned} \tag{4.10}$$

We define the autocovariance matrix (ACM) as:

$$\mathscr{R}(N) = E\begin{bmatrix} C_0 & \cdots & C_{N-1} \\ \vdots & \ddots & \vdots \\ C_{N-1}^T & \cdots & C_0 \end{bmatrix} \tag{4.11}$$

whose elements are derived from data according to

$$\hat{C}_k = \frac{1}{N_d} \sum_{t=k}^{N_d} \mathscr{Z}_t \mathscr{Z}_{t-k}^T \tag{4.12}$$

and N is a user-defined parameter. Using (4.10) and (4.11) the ACM of the L-innovations can be written as:

$$\mathscr{R}(N) = \mathscr{O}_{ALS} M^- \mathscr{O}_{ALS}^T + \Gamma \left[\bigoplus_{i=1}^{N} \bar{G}\bar{Q}_w \bar{G}^T \right] \Gamma^T + \Psi \left[\bigoplus_{j=1}^{N} R_v \right] + \left[\bigoplus_{j=1}^{N} R_v \right] \Psi^T + \left[\bigoplus_{j=1}^{N} R_v \right] \tag{4.13}$$

where,

$$\mathscr{O}_{ALS} = \begin{bmatrix} H \\ H\bar{F} \\ \vdots \\ H\bar{F}^{N-1} \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0 & 0 & 0 & 0 \\ H & 0 & 0 & 0 \\ \vdots & \ddots & & \vdots \\ H\bar{F}^{N-2} & \cdots & H & 0 \end{bmatrix}, \quad \Psi = \Gamma \left[\bigoplus_{j=1}^{N} -FL \right]$$

Also, the covariance of the noise $\bar{w}_t$ is given by

$$E\left[\bar{w}_t (\bar{w}_t)^T\right] = \bar{Q}_w = \begin{bmatrix} Q_w & 0 \\ 0 & R_v \end{bmatrix}$$

In order to show the problem formulation as a Least-Squares problem, equation (4.13) is given in stacked form. Henceforth, $(\cdot)_s$ denotes the outcome to apply the *vec* operator to $(\cdot)$. Equation (4.13) is written in a stacked way using the standard definitions [31] of the Kronecker sum $\oplus$, Kronecker product $\otimes$, and direct sum $\bigoplus$ as:

$$\begin{aligned} [\mathscr{R}(N)]_s &= \left[(\mathscr{O}_{ALS} \otimes \mathscr{O}_{ALS})(I_{n^2} - \bar{F} \otimes \bar{F})^{-1} + (\Gamma \otimes \Gamma)\mathscr{I}_{n,N}\right](G \otimes G)(Q_w)_s \\ &+ \left\{ \left[(\mathscr{O}_{ALS} \otimes \mathscr{O}_{ALS})(I_{n^2} - \bar{F} \otimes \bar{F})^{-1} + (\Gamma \otimes \Gamma)\mathscr{I}_{n,N}\right](FL \otimes FL) + \left[\Psi \oplus \Psi + I_{p^2N^2}\right]\mathscr{I}_{p,N} \right\} (R_v)_s \end{aligned} \tag{4.14}$$

Equation (4.14) can be written as a LS problem, considering that $\mathscr{R}(N)_s$ can be estimated from (4.11) using the acquired data.

Given $\mathscr{A}x = b$, with

$$\begin{aligned} D &= \left[\; (\mathscr{O}_{ALS} \otimes \mathscr{O}_{ALS}(I_{n^2} - \bar{F} \otimes \bar{F})^{-1} + (\Gamma \otimes \Gamma)\mathscr{I}_{n,N}) \; \right] \\ \mathscr{A} &= \left[\; D(G \otimes G) \quad D(FL \otimes FL) + \left[\Psi \oplus \Psi + I_{p^2N^2}\right]\mathscr{I}_{p,N} \; \right] \\ x &= \left[\; (Q_w)_s^T \quad (R_v)_s^T \; \right]^T \\ b &= \mathscr{R}(N)_s \end{aligned}$$

where $\mathscr{I}_{p,N}$ is a permutation matrix to convert the direct sum to a vector, i.e $\mathscr{I}_{p,N}$ is the $(pN)^2 \times p^2$ matrix of zeros and ones satisfying:

$$\left(\bigoplus_{j=1}^{N} R_v\right)_s = \mathscr{I}_{p,N}(R_v)_s$$

We define the ALS estimate as follows:

$$\begin{aligned} \hat{x} &= \arg\min_x \|\mathscr{A}x - b\|_2^2 \\ s.t \;\; & Q_w, R_v \geq 0 \end{aligned} \tag{4.15}$$

in which $\hat{x} = \begin{bmatrix} (Q_w)_s^T & (R_v)_s^T \end{bmatrix}^T$, and $\hat{b} = \hat{\mathscr{R}}(N)_s$. These steps are summarized in **Algorithm 1**.

Algorithm 1 ALS Algorithm

for $j = 1$ to $N-1$ **do**
  $\hat{C}_j = \frac{1}{N_d - j}\sum_{i=1}^{N_d - j} \mathscr{Z}_i \mathscr{Z}_{i+j}^T$
end for
Compute $\hat{b} = \hat{\mathscr{R}}(N)_s$ from Eq. (4.11)
Solve $\begin{bmatrix} \hat{Q}_w \\ \hat{R}_v \end{bmatrix} = \arg\min_{Q_w, R_v} \left\| \mathscr{A} \begin{bmatrix} (Q_w)_s \\ (R_v)_s \end{bmatrix} - \hat{b} \right\|_2^2 \quad \text{s.t.} \quad Q_w \geqslant 0, \quad R_v \geqslant 0$

The existence, uniqueness and unbiasedness nature of the estimates have been proved in [31] and already discussed in deliverable D5.2.

4.3 Recursive ALS with MHE-based innovations

In order to adaptively tune the noise covariance matrices to be used in the MHE algorithms developed in the HD-MPC project, we now propose a new adaptive ALS scheme to be used in on-line operations. It is based on the adaptive update of matrices Q_w and R_w used in the estimation problem (4.7). The update of Q_w and R_w is carried out, at each time instant, according to the equations

$$\begin{aligned} Q_w^+ &= Q_w + \rho_Q(Q_w^{opt} - Q_w) \\ R_v^+ &= R_v + \rho_R(R_v^{opt} - R_v) \end{aligned} \tag{4.16}$$

where Q_w^{opt} and R_v^{opt} result from (4.15). In turn, the terms $\hat{C}_k$, used in (4.16), are computed and updated, at each time instant $t > k$, using a recursive (simplified) version of (4.12), i.e.

$$\hat{C}_k^{(t)} = \hat{C}_k^{(t-1)} + \rho_C(\mathscr{Z}_t \mathscr{Z}_{t-k}^T - \hat{C}_k^{(t-1)}) \tag{4.17}$$

In equations (4.16) and (4.17), the parameters ρ_Q, ρ_R, and ρ_C are suitably defined parameters taking values in the interval $(0,1)$.
The new algorithm (i.e., Algorithm 2) for covariance estimation is more formally reported below, where N is the MHE time window, N_{sim} is the simulation time, N_{ALS} is the ALS tuning parameter, t_{start} is the time at which the adaptation starts, Q_{ini} and R_{ini} are the initial guesses for the covariances.

Algorithm 2 Adaptive Autocovariance Least Squares (AALS)

Define N, N_{sim}, N_{ALS}, t_{start}, Q_{ini}, R_{ini}
Set $Q_w = Q_{ini}$ and $R_v = R_{ini}$
Compute Π_{i-N} as the stationary prediction error covariance matrix through the algebraic Riccati equation (ARE).
Initialize the sampled covariances $\hat{C}_k$ ($k = 0, \ldots, N_{ALS} - 1$).
for $t = N + 1$ to $N_{sim} - 1$ **do**
  Solve the MHE problem (4.7).
  if $i \geq N_{ALS} + 1$ **then**
    Update the sampled covariances $\hat{C}_k$ ($k = 0, \ldots, N_{ALS} - 1$) using (4.17).
  end if
  if $t > t_{start}$ **then**
    Solve the ALS problem (4.15).
    Update the estimated covariances Q_w and R_v using (4.16).
    Update Π_{i-N}.
  end if
end for

4.4 Case studies

4.4.1 Van der Vusse reaction system

A non-isothermal Van der Vusse reactor system has been considered to test the adaptive algorithm; this system has already been used as a benchmark in several control and estimation contributions [1, 9, 11, 32].

The reactor is a vessel where an exothermic reaction is given; the excess of heat is removed by means of a cold flow through a jacket, making this flow rate critical in order to obtain the desired amount of product. From mass and energy balance equations it is possible to derive the dynamic 4-th order model of the system [11]:

$$\begin{aligned}
\dot{c}_A &= \frac{F}{V_R}(c_{A0} - c_A) - k_1(T)c_A - k_3(T)c_A^2 \\
\dot{c}_B &= -\frac{F}{V_R}c_B + k_1(T)c_A - k_2(T)c_B \\
\dot{T} &= -\frac{1}{\rho C_p}(k_1(T)c_A\Delta H_{r1} + k_2(T)c_B\Delta H_{r2} + k_3(T)c_A^2\Delta H_{r3}) + \frac{F}{V_R}(T_0 - T) + \frac{k_w A_R}{\rho C_p V_R}(T_j - T) \\
\dot{T}_j &= \frac{1}{m_j C_{pj}}(\dot{Q}_j + k_w A_R(T - T_j))
\end{aligned} \tag{4.18}$$

where c_A and c_B are the concentrations of components A and B in the effluent stream, respectively, T is the reaction temperature, and T_j is the coolant temperature, c_{A0} is the concentration of A in the inlet stream, F is the feed flow rate, V_R is the reactor volume and $\dot{Q}_j$ is the rate of heat addition or removal. The reaction coefficients k_i, $i = 1, \ldots, 3$ are given by means of the Arrhenius equation:

$$k_i(T) = k_{0i}e^{E_i/T} \tag{4.19}$$

Table 4.1: Parameters of the Van der Vusse reactor

Symbol	Value	Symbol	Value
$k_{01}\left[h^{-1}\right]$	1.287×10^{12}	$\Delta H_{r1}\,[kJ/mol]$	4.2
$k_{02}\left[h^{-1}\right]$	1.287×10^{12}	$\Delta H_{r2}\,[kJ/mol]$	-11
$k_{03}\,[l/(mol\cdot h)]$	9.043×10^{9}	$\Delta H_{r1}\,[kJ/mol]$	-41.85
$E_1\,[K]$	-9758.3	$\rho\,[kg/l]$	0.9342
$E_2\,[K]$	-9758.3	$C_p\,[kJ/(kg\cdot K)]$	3.01
$E_3\,[K]$	-8560	$k_w\left[kJ/(h\cdot m^2\cdot K)\right]$	4032
$A_R\left[m^2\right]$	0.215	$V_R\left[m^3\right]$	0.01
$m_j\,[kg]$	5	$C_{pj}\,[kJ/(kg\cdot K)]$	2

Table 4.2: Nominal steady-state values for the Van der Vusse reactor

Symbol	Value	Symbol	Value
$c_A^s\,[mol/l]$	2.1402	$F^s\,[l/h]$	141.9
$c_B^s\,[mol/l]$	1.0903	$\dot{Q}_j^s\,[kJ/h]$	-1113.5
$T^s\,[K]$	387.34	$c_{A0}^s\,[mol/l]$	5.1
$T_j^s\,[K]$	386.06	$T_0^s\,[K]$	378.05

The model parameters are given in Table 4.1.

A linear, time-invariant, discrete-time model has been obtained by linearizing (4.18) around the operating point presented in Table 4.2, and discretizing it with a sampling time $T_s = 0.01h$.

$$\begin{aligned} x_{t+1} &= A_d x_t + B_d u_t + B_{p,d} p_t + \sigma_d w_t \\ y_t &= C_d x_t + v_t \end{aligned} \tag{4.20}$$

where

$$x = \begin{bmatrix} c_A - c_A^s \\ c_B - c_B^s \\ T - T^s \\ T_j - T_j^s \end{bmatrix}, \quad u = \begin{bmatrix} F - F^s \\ \dot{Q}_j - \dot{Q}_j^s \end{bmatrix}, \quad p = \begin{bmatrix} c_{A0} - c_{A0}^s \\ T_0 - T_0^s \end{bmatrix} \tag{4.21}$$

$v_k \sim N(0, R_v)$ and $w_k \sim N(0, Q_w)$ are the noises disturbing the process and the measurements, respectively, and $\sigma_d = \text{diag}\left(\left[c_A^s, c_B^s, T^s, T_j^s\right]\right)$. Moreover, we assume that c_B and T are the measured variables, namely

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

For the sake of simplicity, we assume the covariance matrices are diagonal matrices (we take this as a further constraint in the optimization problem (4.15)):

$$Q_w = qI, \quad R_v = rI \tag{4.22}$$

where I are identity matrices of proper dimension. In the data generation process, the real values of q and r are $q^o = 0.8$ and $r^o = 0.2$.

Table 4.3: AALS parameters for the Van der Vusse reactor example

Parameter	Value	Parameter	Value
N	10	N_{sim}	6000
N_{ALS}	10	t_{start}	1000
q^o	0.8	r^o	0.2
q_{ini}	3	r_{ini}	0.06
ρ	0.001		

The adaptive ALS algorithm has been used with the parameters specified in Table 4.3, being q_{ini} and r_{ini} the initial guesses for q and r respectively. Figure 4.1 shows the real and estimated values of the parameters q and r. Moreover, in Figure 4.2 the true and estimated values of the state variable c_A are reported: it is clear that the estimate is progressively improved as the covariance matrices, i.e. the weights in the MHE problem, are more precisely evaluated.

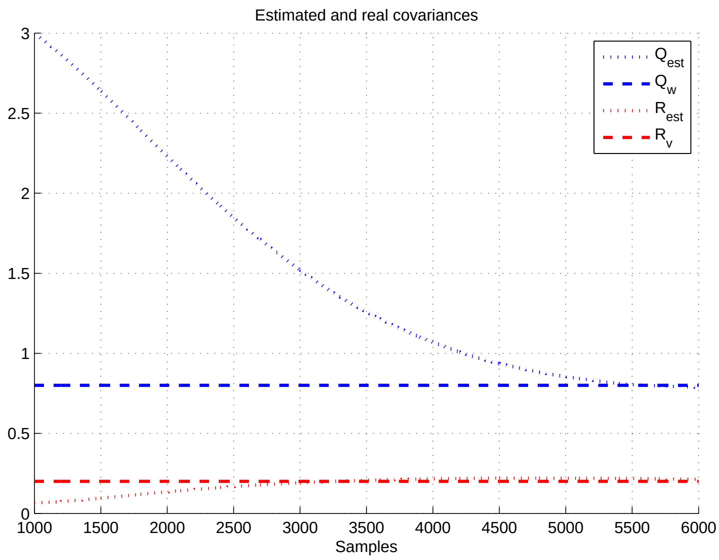

Figure 4.1: Convergence of the covariances via the adaptive law.

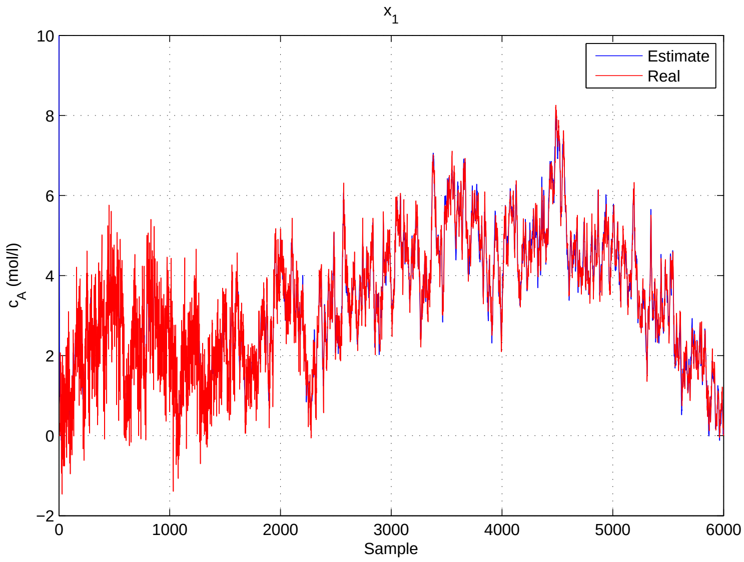

Figure 4.2: True and estimated values of c_A with the adaptive tuning of the covariances.

4.4.2 Mehra's example

As a second test case, consider the example presented in [28], i.e. the linear time-invariant discrete-time system:

$$F = \begin{bmatrix} 0.75 & -1.74 & -0.3 & 0 & -0.15 \\ 0.09 & 0.91 & -0.0015 & 0 & -0.008 \\ 0 & 0 & 0.95 & 0 & 0 \\ 0 & 0 & 0 & 0.55 & 0 \\ 0 & 0 & 0 & 0 & 0.905 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 24.64 & 0 & 0 \\ 0 & 0.835 & 0 \\ 0 & 0 & 1.83 \end{bmatrix}$$
$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \qquad (4.23)$$

The data are generated according to the following distributions:

$$w(k) \sim N(0, 0.5 I_3), \qquad v(k) \sim N(0, I_2), \qquad (4.24)$$

The parameters used in the adaptive tuning algorithm are presented in Table 4.4, where q_{ini} and r_{ini} are the initial guesses for $q^o = 0.5$ and $r^0 = 1$ respectively. Figure 4.3 shows the outcome of the Algorithm with $\rho = 0.001$; this choice guarantees a smooth and slow convergence of q and r. As expected, using the value $\rho = 0.1$, the convergence is faster, but with an oscillatory response, see Figure 4.4.

Also in this example, it is apparent that the adaptive tuning algorithm performs very well, so that a significant improvement of any estimation algorithm requiring the knowledge of the noise covariances can be achieved.

Table 4.4: AALS parameters

Parameter	Value	Parameter	Value
N	10	N_{sim}	6000
N_{ALS}	10	t_{start}	1000
q^o	0.5	r^o	1
q_{ini}	4	r_{ini}	0.2
ρ	0.001		

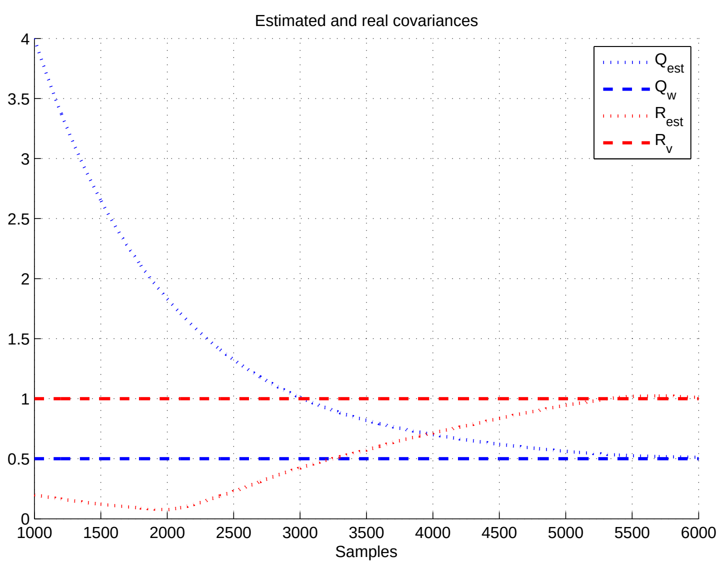

Figure 4.3: Convergence of the covariances with the adaptive law and $\rho = 0.001$.

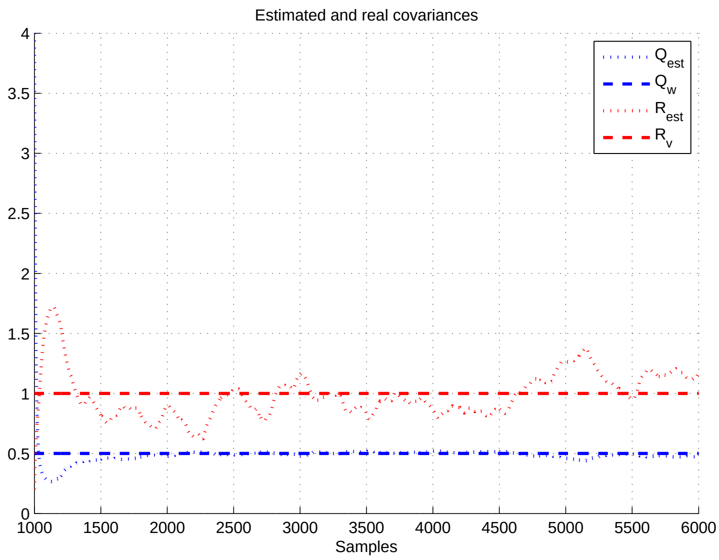

Figure 4.4: Convergence of the covariances with the adaptive law and $\rho = 0.1$.

Bibliography

[1] B. M. Åkesson, J. B. Jørgensen, N. K. Poulsen, and S. B. Jørgensen, "A Generalized Autocovariance Least-Squares Method for Kalman Filter Tuning". *Journal of Process Control*, **18**, pp: 769-779, 2008.

[2] A. Alessandri, M. Baglietto, and G. Battistelli. Receding-horizon estimation for discrete-time linear systems. *IEEE Trans. on Automatic Control*, 48(3):473 – 478, 2003.

[3] A. Alessandri, M. Baglietto, and G. Battistelli. Moving horizon state estimation for nonlinear discrete-time systems: new stability results and approximation schemes. *Automatica*, 44:1753 – 1765, 2008.

[4] A. Alessandri, M. Baglietto, T. Parisini, and R. Zoppoli. A neural state estimator with bounded errors for nonlinear systems. *IEEE Trans. on Automatic Control*, 44(11):2028 – 2042, 1999.

[5] P. Alriksson and A. Rantzer. Distributed Kalman filtering using weighted averaging. In *Proc. of the 17th International Symposium on Mathematical Theory of Networks and Systems*, Kyoto, Japan, 2006.

[6] D. Angeli. A Lyapunov approach to incremental stability properties. *IEEE Trans. on Automatic Control*, 47:410 – 421, 2002.

[7] B. Bollobás. *Modern graph theory*, volume 184 of *Graduate texts in mathematics*. Springer, New York, 1998.

[8] R. Carli, A. Chiuso, L. Schenato, and S. Zampieri. Distributed Kalman filtering based on consensus strategies. *IEEE Journal on Selected Areas In Communications*, (4):622 – 633, 2008.

[9] H. Chen, A. Kremling, and F. Algöwer, "Nonlinear Predictive Control in a nonlinear CSTR". In Proc. of ECC, pp.3247-3252, 1995.

[10] S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. An ISS small gain theorem for iss general networks. *Mathematics of Control, Signals, and Systems*, 19(2):93–122, 2007.

[11] S. Engell, and K. U. Klatt, "Nonlinear Control of a nonminimum phase CSTR". In Proc. of ACC, pp.2941-2945, 1993.

[12] D. Faille. Control specification for hydro power valleys. HD-MPC STREP deliverable D7.2.1. `http://www.ict-hd-mpc.eu/deliverables/hd_mpc_D_7 _2_1.pdf`, 2009.

[13] M. Farina, G. Ferrari-Trecate, and R. Scattolini. Distributed moving horizon estimation for sensor networks. In *Proc. 1st IFAC Workshop on Estimation and Control of Networked Systems, Venice, Italy*, pages 126 – 131, 2009.

[14] M. Farina, G. Ferrari-Trecate, and R. Scattolini. A moving horizon scheme for distributed state estimation. In *48th IEEE Conference on Decision and Control and 28th Chinese Control Conference, Shanghai, China*, pages 1818 – 1823, 2009.

[15] M. Farina, G. Ferrari-Trecate, and R. Scattolini. Distributed moving horizon estimation for linear constrained systems. *IEEE Trans. on Automatic Control*, Volume 55 (11), pages 2462-2475, 2010.

[16] M. Farina, G. Ferrari-Trecate, and R. Scattolini. Distributed moving horizon estimation for nonlinear constrained systems. In *8th IFAC Symposium on Nonlinear Control Systems*, Bologna, Italy, 2010.

[17] M. Farina, G. Ferrari-Trecate, and R. Scattolini. Moving horizon state estimation of large-scale constrained partitioned systems. *Automatica*, 46(5):910 – 918, 2010.

[18] M. Farina, G. Ferrari-Trecate, and R. Scattolini. Distributed moving horizon estimation for nonlinear constrained systems. 2010. *Int. Journal of Robust and Nonlinear Control*, 2011. In press.

[19] G. Ferrari-Trecate, D. Mignone, and M. Morari. Moving horizon estimation for hybrid systems. *IEEE Trans. on Automatic Control*, 47(10):1663 – 1676, Oct. 2002.

[20] G. C. Goodwin, M. M. Seron, and J. A. De Doná. *Constrained Control and Estimation*. Springer, New Jersey, 2005.

[21] E. Gullhamn. *Control of Water Content and Retention in Hydropower Plant Cascades*. PhD thesis, KTH Computer Science and Communication, Stockholm, Sweden, 2004.

[22] H. R. Hashemipour, S. Roy, and A. J. Laub. Decentralized structures for parallel Kalman filtering. *IEEE Trans. on Automatic Control*, 33(1):88 – 94, Jan. 1988.

[23] K. Holmström. The TOMLAB optimization environment in MATLAB. *Advanced Modeling and Optimization*, pages 47–69, 1999.

[24] M. Kamgarpour and C. Tomlin. Convergence properties of a decentralized Kalman filter. *Proc.* 47^{th} *IEEE Conference on Decision and Control*, pages 3205 – 3210, 2008.

[25] U. A. Khan and J. M. F. Moura. Distributing the Kalman filter for large-scale systems. *IEEE Trans. on Signal Processing*, 56(10):4919 – 4935, Oct. 2008.

[26] X. Litrico and V. Fromion. Infinite dimensional modelling of open-channel hydraulic systems for control purposes. In *Proc.* 41^{st} *IEEE Conference on Decision and Control*, pages 1681–1686, 2002.

[27] P. O. Malaterre and J. P. Baume. Modeling and regulation of irrigation canals: existing applications and ongoing researches. In *Proc. International Conference on Systems, Man, and Cybernetics*, pages 3850–3855, 1998.

[28] R. K. Mehra, "On the Identification of Variances and Adaptive Kalman Filtering". *IEEE Trans. on Autom. Control*, **AC-15**, No 2, pp: 175-184, 1970.

[29] R. K. Mehra, "Approaches to adaptive filtering". *IEEE Trans. on Autom. Control*, **17**, No 17, pp: 903-908, 1972.

[30] A. G. O. Mutambara. *Decentralized estimation and control for multisensor systems*. CRC Press, 1998.

[31] B. J. Odelson, M. R. Rajamani, and J. B. Rawlings, "A New Autocovariance Least-Squares Method for Estimating Noise Covariances". *Automatica*, **42**, pp: 303-308, 2006.

[32] B. A. Ogunnaike and W. H. Ray, "Process Dynamics, Modelling, and Control. ". Oxford University Press, New York, 1994.

[33] R. Olfati-Saber. Distributed Kalman filter with embedded consensus filters. *Proc.* 44^{th} *IEEE Conference on Decision and Control - European Control Conference*, pages 8179 – 8184, 2005.

[34] R. Olfati-Saber. Distributed Kalman filtering for sensor networks. *Proc.* 46^{th} *IEEE Conference on Decision and Control*, pages 5492 – 5498, 2007.

[35] R. Olfati-Saber. Kalman-consensus filter: Optimality, stability and performance. *Proc.* 48^{th} *IEEE Conference on Decision and Control*, pages 7036 – 7042, 2009.

[36] R. Olfati-Saber and J. Shamma. Consensus filters for sensor networks and distributed sensor fusion. *Proc.* 44^{th} *IEEE Conference on Decision and Control - European Control Conference*, pages 6698 – 6703, 2005.

[37] B. S. Rao and H. F. Durrant-Whyte. Fully decentralised algorithm for multisensor Kalman filtering. In *IEE Proc. on Control Theory and Applications, D*, volume 138, pages 413 – 420, Sept. 1991.

[38] C. V. Rao and J. B. Rawlings. Nonlinear moving horizon state estimation. *in F. Allgöwer and A. Zheng, editors*, Nonlinear Model Predictive Control, *Progress in Systems and Control Theory, Birkhauser*, pages 45–70, 2000.

[39] C. V. Rao, J. B. Rawlings, and J. H. Lee. Stability of constrained linear moving horizon estimation. *Proc. American Control Conference*, pages 3387 – 3391, 1999.

[40] C. V. Rao, J. B. Rawlings, and J. H. Lee. Constrained linear state estimation - a moving horizon approach. *Automatica*, 37:1619 – 1628, 2001.

[41] C. V. Rao, J. B. Rawlings, and D. Q. Mayne. Constrained state estimation for nonlinear discrete-time systems: Stability and moving horizon approximations. *IEEE Trans. on Automatic Control*, 48(2):246 – 258, 2003.

[42] C. V. Rao. *Moving Horizon Strategies for the Constrained Monitoring and Control of Nonlinear Discrete-Time Systems*. PhD thesis, University of Winsconsin-Madison, 2000.

[43] J. B. Rawlings and D. Q. Mayne. *Model predictive control: theory and design*. Nob Hill Publishing, LLC, 2009.

[44] P. Rostalski, G. Papafotiou, C. Setz, A. Heinrich, and M. Morari. Application of model predictive control to a cascade of river power plants. In *Proceedings of the 17th World Congress*, pages 11876 - 11983, Seoul, Corea, 2008.

[45] N. R. Sandell Jr., P. Varaiya, M. Athans, and M. Safonov. Survey of decentralized control methods for large scale systems. *IEEE Trans. on Automatic Control*, 23(2):108 – 128, Apr 1978.

[46] R. Scattolini. Architectures for distributed and hierarchical model predictive control - a review. *Journal of Process Control*, 19:723–731, 2009.

[47] D. P. Spanos, R. Olfati-Saber, and R. M. Murray. Approximate distributed Kalman filtering in sensor networks with quantifiable performance. *Fourth International Symposium on Information Processing in Sensor Networks*, pages 133 – 139, 2005.

[48] D. D. Šiljac. *Large-scale dynamic systems. Stability and structure.* North Holland, 1978.

[49] S. S. Stanković, M. S. Stanković, and D. M. Stipanović. Consensus based overlapping decentralized estimation with missing observations and communication faults. *Automatica*, 45:1397 – 1406, 2009.

[50] S. S. Stanković, M. S. Stanković, and D. M. Stipanović. Consensus based overlapping decentralized estimator. *IEEE Trans. on Automatic Control*, 54(2):410 – 415, Feb. 2009.

[51] R. Vadigepalli and F. J. Doyle III, A distributed state estimation and control algorithm for plantwide processes. *IEEE Trans. on Control Systems Technology*, 11(1):119 – 127, Jan. 2003.