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Executive summary

In this report we address and solve different sorts of robust optimal control problems. First, we introduce an approach to uncertain nonlinear systems, where the uncertainty enters the dynamics in an affine fashion. If this holds, one may eliminate the classical min-max structure from the robust dynamic optimization problem.
Second, we discuss a novel technique to provide robust linear control laws to general nonlinear dynamic systems.
Third, we address a robust controller design problem for distributed linear systems using duality of linear programming.

Chapter 1

Synopsis of the report

In this report we are concerned with robust optimal control problems. In Chapter 2 and Chapter 3 we discuss different solution methods to the general problem

$$\min_{x,u} \; f_0(x,u) \tag{1.1}$$

$$\text{s.t. } f_i(x,u) \le 0 \quad i = 1,\dots,n_f \tag{1.2}$$

$$g_j(x,u,p) = 0 \quad j = 1,\dots,n_x, \tag{1.3}$$

where x, u and p denote the states, controls and uncertain parameters of the underlying optimal control problem, respectively. We define the worst-case scenario by introducing a multi-level optimization problem. In this framework two players, the robust controller and the uncertainty, compete with each other. The adverse player (uncertainty) wants to violate the constraints, while respecting the system dynamics

$$\phi_i(u) := \begin{pmatrix} \max_{x,p} & f_i(x,u) \\ \text{s.t.} & g(x,u,p) = 0 \\ & \|p - \overline{p}\| \le 1 \end{pmatrix}. \tag{1.4}$$

The controller wants to respect the constraints by choosing an u such that

$$\min_{u} \; \phi_0(u) \tag{1.5}$$

$$\text{s.t. } \phi_i(u) \le 0 \quad i = 1,\dots,n_f. \tag{1.6}$$

In Chapter 2 we propose an algorithm to solve a special case of (1.4) that is

$$\phi_i(u) := \begin{pmatrix} \max_{x,p} & f_i(x,u) \\ \text{s.t.} & A(u)x + B(u)p + b(u) = 0 \\ & \|p - \overline{p}\| \le 1 \end{pmatrix}, \; i = 1,\dots,n_f, \tag{1.7}$$

where $\overline{p}$ corresponds to the nominal control case. Note that the uncertainty enters the dynamic system in an affine manner. In such cases the linearization of $g(.)$ is exact, which results in numerically tractable dynamic optimization problem and allows for a more conservative controller.

In Chapter 3 we present techniques to solve robust optimal control problems for nonlinear dynamic systems in a conservative approximation. Here, we assume that the nonlinear dynamic system is affected by a time-varying uncertainty whose L-infinity norm is known to be bounded. By employing specialized explicit upper estimates for the nonlinear terms in the dynamics we propose a strategy to design a linear control law which guarantees that given constraints on the states and controls are robustly satisfied when running the system in closed- loop mode. Finally, the mathematical techniques are illustrated by applying them to a tutorial example.

In Chapter 4 a decentralized robust control scheme is considered. We formulate a robust feasibility problem for the design of a reference governor to provide set-points for the lower- level control in a two-layer hierarchical system. We regard the linear state-space model

$$x(k+1) = Ax(k) + Br(k) + Gd(k) \tag{1.8}$$

$$z(k) = Cx(k) \tag{1.9}$$

with the set-point deviation $r(k)$ and disturbance perturbation $d(k)$ and system output $z(k)$. Using linear programming duality, solutions to the robust feasibility problem (i.e. both necessary and sufficient conditions for the existence of an admissible reference) are given. Three cases are considered: 1) Fixed reference; 2) feedforward management; and 3) affine feedback management. The computationally efficient results can be implemented in supervisory control in SCADA networks.

Chapter 2

Robust nonlinear optimal control of dynamic systems with affine uncertainties

In this chapter we are interested in the robust optimization of open-loop controlled uncertain systems that are linear in the uncertain states and disturbances but possibly nonlinear in the remaining states and the control input. Here, the main challenge is to robustly regard inequality state constraints. For this aim we start by introducing Lyapunov differential equations as a well-known tool [16, 5, 12, 13, 4] to compute variance-covariance matrix functions. We include results from [8] without proofs.

2.1 Uncertain linear time-varying systems

Let us consider a linear time-varying system with a differential state vector $x : \mathbb{R} \to \mathbb{R}^{n_x}$ that is excited by a disturbance function $w : \mathbb{R} \to \mathbb{R}^{n_w}$:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)w(t) \\ y(t) &= C(t)x(t) \\ x(0) &= B_0 w_0 \end{aligned} \tag{2.1}$$

with $t \in \mathbb{T}$ and $\mathbb{T} := [0,T] \subset \mathbb{R}$, where T is the length of the time horizon. Here, we assume that the coefficients $A : \mathbb{T} \to \mathbb{R}^{n_x \times n_x}$, $B : \mathbb{T} \to \mathbb{R}^{n_x \times n_w}$, and $C : \mathbb{T} \to \mathbb{R}^{n_y \times n_x}$ are square-integrable functions on the finite interval $\mathbb{T}$. In addition, $y : \mathbb{T} \to \mathbb{R}^{n_y}$ is called the output function while $w_0 \in \mathbb{R}^{n_{w_0}}$ is an uncertainty affecting the initial state via the matrix $B_0 \in \mathbb{R}^{n_x \times n_{w_0}}$. At this point, it is worth to mention that we will later also regard a time dependent function v to be optimized which might enter A, B, and C in a possibly nonlinear way, but for the moment we suppress this dependence on v to achieve a convenient notation.

The fundamental solution $G : \mathbb{T} \times \mathbb{T} \to \mathbb{R}^{n_x \times n_x}$ of (2.1) is the unique solution of the initial value problem:

$$\frac{\partial G(t,\tau)}{\partial t} = A(t)G(t,\tau) \quad \text{with} \quad G(\tau,\tau) = \mathbb{I} \tag{2.2}$$

for all $t, \tau \in \mathbb{T}$. Using this notation, we can write the output function y in the form [19]

$$y(t) = H_t^0 w_0 + \int_0^T H_t(\tau) w(\tau) \mathrm{d}\tau \tag{2.3}$$

with the Green's or impulse response function $H_t : \mathbb{T} \to \mathbb{R}^{n_y \times n_w}$ being defined by

$$\forall t, \tau \in \mathbb{T} : \; H^t(\tau) := \begin{cases} C(t)G(t,\tau)B(\tau) & \text{if } \tau \leq t \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

and $H_t^0 := C(t)G(t,0)B_0$. Obviously, the differential equation (2.2) for the fundamental solution G is completely independent of the matrix functions B and C. In order to describe the dependence of the state on the disturbance $\omega := (w_0, w(\cdot))$, we make use of Lyapunov differential equations of the form

$$\begin{array}{rcl} \dot{P}(t) & = & A(t)P(t) + P(t)A(t)^T + B(t)B(t)^T \\ P(0) & = & B_0 B_0^T \end{array} . \tag{2.5}$$

The state of this differential equation is a matrix valued function $P : \mathbb{T} \to \mathbb{R}^{n_x \times n_x}$. Note that P propagates the following important information [12, 13, 4]: if the disturbance w entering the system (2.1) is a Gaussian white noise process with

$$\begin{array}{l} \mathbb{E}\{w(t)\} = 0 \quad \text{and} \\ \mathbb{E}\{w_i(t_1) w_j(t_2)^T\} = \Sigma(t_1)\, \delta(t_2 - t_1)\, \delta_{i,j} \end{array}$$

for all $t, t_1, t_2 \in \mathbb{T}$ and for all $i, j \in \{1, .., n_w\}$ while the uncertain value w_0 is Gaussian distributed with variance-covariance matrix $\mathbb{I} \in \mathbb{R}^{n_x \times n_x}$ then $P(t)$ is the variance-covariance matrix of the state $x(t)$ for all times $t \in \mathbb{T}$. Consequently, the variance-covariance matrix of the output y is at each time $t \in \mathbb{T}$ given by $C(t)P(t)C(t)^T$.

2.2 Inequality constrained Lyapunov differential equations

As almost all systems in the real-world are subject to inequality constraints we like to discuss the linear system (2.1) in combination with output constraints of the form

$$\forall t \in \mathbb{T}, \forall i \in \{1, \ldots, n_y\} : \quad y_i(t) \leq d_i(t) \, . \tag{2.6}$$

The function $d : \mathbb{T} \to \mathbb{R}^{n_y}$ is assumed to be continuous with strictly positive components. Furthermore, $C_i(t)$ denotes the i-th row vector of $C(t)$. For all theoretical purposes in this section, we will assume $d_i(t) = 1$ by rescaling the rows $C_i(t)$ for all components $i \in I := \{1, \ldots, n_y\}$. In this case, all constraints have the simple form $y_i(t) \leq 1$.

Now, we suggest to consider a corresponding constrained Lyapunov differential equation for the matrix valued function P defined by

$$\begin{array}{rcl} \dot{P}(t) & = & A(t)P(t) + P(t)A(t)^T + B(t)B(t)^T \\ 0 & \geq & C_i(t)P(t)C_i(t)^T - 1 \\ P(0) & = & B_0 B_0^T \, . \end{array} \tag{2.7}$$

The aim of this section is to discuss a worst case interpretation of the constrained Lyapunov system (2.7) based on the assumption that the disturbance $\omega := (w_0, w(\cdot))$ is a bounded but unknown function. For this aim, we consider the Hilbert space L_2 of all square-integrable functions from $\mathbb{T}$ to $\mathbb{R}^{n_w}$. Moreover, we define an inner product $\langle \cdot \mid \cdot \rangle_W : W \times W \to \mathbb{R}$ in the space $W := \mathbb{R}^{n_{w_0}} \times L_2$ and the corresponding W-norm $\| \cdot \|_W : W \to \mathbb{R}$ by

$$\begin{array}{rcl} \langle \omega_1 \mid \omega_2 \rangle_W & := & w_{0,1}^T w_{0,2} + \int_0^T w_1(\tau)^T w_2(\tau)\, \mathrm{d}\tau \quad \text{and} \\ \|\omega_1\|_W & := & \sqrt{\langle \omega_1, \omega_1 \rangle_W} \end{array} \tag{2.8}$$

for all $\omega_1, \omega_2 \in W$ and $\omega_j = (w_{0,j}, w_j(\cdot))$ for $j \in \{1,2\}$.

In the next step, we define the ball $\mathscr{B} \subseteq W$ by

$$\mathscr{B} := \{\omega \in L_2 \mid \|\omega\|_W \leq 1\} \; .$$

The worst case interpretation of the constrained Lyapunov differential equation can now be stated as follows:

Theorem 2.2.1 *The constraints of the form $y_i(t) \leq 1$ are satisfied for all times $t \in \mathbb{T}$, all components $i \in I$, and all disturbances $\omega \in \mathscr{B}$ if and only if the constrained Lyapunov differential equation* (2.7) *admits a feasible solution P.*

Remark:

- Provided that w is a unit-bandwidth white noise process, while w_0 is a Gaussian distributed random variable with variance-covariance matrix $\mathbb{I}$, we have

  $$\begin{array}{l} \forall t \in \mathbb{R}, \, i \in I : \\ \text{Probability}\, \{\, y_i(t) > 1 \,\} < \int_1^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \, \mathrm{d}z \, . \end{array}$$

  if and only if the constrained Lyapunov differential equation (2.7) admits a feasible solution P. This stochastic single chance interpretation can directly be seen by making use of the fact that we have $\mathbb{E}\left\{y_i(t)^2\right\} = C_i(t)P(t)C_i(t)^T$ in this case.

Let us transfer the results from the previous section to uncertain optimal control problems of the form

$$\boxed{\begin{array}{rrcl} \min\limits_{x(\cdot),v(\cdot),T} & & & J[v(\cdot),T] \\ \text{s.t.} & & & \\ \forall t \in \mathbb{T}: & \dot{x}(t) & = & A(v(t))x(t) + B(v(t))w(t) \\ & & & +r(v(t)) \\ \forall t \in \mathbb{T}: & 0 & \geq & C(v(t))x(t) - d(v(t)) \\ & x(0) & = & r_0(v(0)) + B_0 w_0 \, , \\ & v & \in & \mathbb{V} \end{array}} \; . \tag{2.9}$$

Here, $v : \mathbb{R} \to \mathbb{R}^{n_v}$ is the nonlinear behavior of the system. This behavior is assumed to lie in a given set $\mathbb{V} \subset L_2$ of measurable functions on the time interval $\mathbb{T}$. In the context of optimal control problems, $\mathbb{V}$ has typically the form

$$\mathbb{V} = \left\{ v : \mathbb{T} \to \mathbb{R}^{n_v} \;\middle|\; \begin{array}{lcl} \forall t \in \mathbb{T}: & & \\ 0 & = & F(t,\dot{v},v) \\ 0 & = & G(v(0),v(T)) = 0 \\ 0 & \geq & H(v(t)) \end{array} \right\} \; . \tag{2.10}$$

Note that this set $\mathbb{V}$ can for example in a trivial case contain a constant parameter which could be formulated as

$$\mathbb{V}_{\text{example}} = \{v : \mathbb{T} \to \mathbb{R}^{n_v} : \dot{v} = 0\} \; ,$$

or piecewise constant controls, or non-trivial implicit differential algebraic equations etc.. However, $\mathbb{V}$ should not depend on ω - otherwise, the system is in general not affine in the uncertainty.

In our formulation, the objective functional J is assumed to be independent of x.[1] The coefficients A, B, C, d are now regarded as functions in v. Moreover, we have introduced a continuous reference function $r : \mathbb{R}^{n_v} \rightarrow \mathbb{R}^{n_x}$, which can, analogous to the nominal initial value $r_0 : \mathbb{R}^{n_v} \rightarrow \mathbb{R}^{n_x}$, also depend on the behaviour v. To formulate the robust counterpart formulation of the above optimization problem we follow the classical approach [3], i.e. we consider the optimization problem

$$\boxed{\begin{array}{lcll} \underset{v(\cdot),T}{\min} & & & J[v(\cdot),T] \\ \text{s.t.} & & & \\ \forall t \in \mathbb{T}: & 0 & \geq & y^{\max}(t\,;v(\cdot),T) - d(v(t)) \\ & v & \in & \mathbb{V} \end{array}}\ , \qquad (2.11)$$

where $y_i^{\max}(t\,;v(\cdot),T)$ is for each $t \in \mathbb{T}$ and each $i \in I$ defined to be the optimal value of the sub-maximization problem

$$\boxed{\begin{array}{lcll} \underset{x(\cdot),w_0,w(\cdot)}{\max} & & & C_i(v(t))x(t) \\ \text{s.t.} & & & \\ \forall \tau \in [0,t]: & \dot{x}(\tau) & = & A(v(\tau))x(\tau) + B(v(\tau))w(\tau) \\ & & & +r(v(\tau)) \\ & x(0) & = & r_0(v(0)) + B_0 w_0 \\ & \omega & \in & \mathscr{B} \end{array}}\ .$$

One of the key results is that the robust counterpart problem (2.11) can, as a direct consequence of Theorem 2.2.1, equivalently be written as

$$\boxed{\begin{array}{lcll} \underset{x_r(\cdot),P(\cdot),v(\cdot),T}{\min} & & & J[v(\cdot),T] \\ \text{s.t.} & & & \\ \forall t \in \mathbb{T}: & \dot{x}_r(t) & = & A(v(t))x_r(t) + r(v(t)) \\ & x_r(0) & = & r_0(v(0)) \\ \forall t \in \mathbb{T}: & \dot{P}(t) & = & A(v(t))P(t) + P(t)A(v(t))^T \\ & & & +B(v(t))B(v(t))^T \\ & P(0) & = & B_0 B_0^T \\ \forall t \in \mathbb{T}, & & & \\ \forall i \in I: & 0 & \geq & C_i(v(t))x_r(t) - d_i(v(t)) \\ & & & +\sqrt{C_i(v(t))P(t)C_i(v(t))^T} \\ & v & \in & \mathbb{V} \end{array}}\ . \qquad (2.12)$$

Note that in this robust counterpart formulation, x_r denotes the reference state trajectory, which would be obtained for $\omega = 0$.

[1]Note that an affine dependence of J on x can always be eliminated by an introduction of slack parameters.

An interesting point about this robust counterpart formulation is that we regard an infinite dimensional disturbance, represented by $\omega \in W$, while there is also an infinite dimensional number of constraints robustly satisfied as the linear path constraints have to be satisfied for *all t* in the time interval $\mathbb{T}$. Note that this is a main advantage in comparison to other existing robust counterpart formulations for optimal control problems, which have been proposed in [6, 17].

Finally, we note that optimal control problems which are affine in the uncertainty can often be found in practice. Even if a problem is not in this form, we might be in one of the following situations:

- If we consider a nonlinear dynamic system, the system can at least for small disturbances be robustified in a linear approximation as proposed in [17, 6]. In this case, the reference trajectory is the solution of the nonlinear dynamic system for $\omega = 0$, which can be optimized, too. In this case, we do not have many guarantees for larger disturbances but for sufficiently small disturbances the linear approximation is valid.

- If we have a linear system of the form

  $$\dot{x}(t) = \widetilde{A}(t)x(t) + B(t)w(t) + D(t)u(t) + r(t)$$

  with a control function u we might be interested in a linear feedback law of the form $u(t) := u_{\mathrm{ref}}(t) + K(t)x(t)$, where $K(t)$ is a matrix function that should be optimized in such a way that the constraints are robustly satisfied. Now, we can summarize the components of K in a vector valued function $v := \mathrm{vec}(K)$. If we define

  $$A(v(t)) := \widetilde{A}(t) + D(t)K(t)$$

  we can obviously transfer the above robust counterpart formulation. Note that this approach can also take linear control constraints into account. In addition, it can be generalized for the case that a linear estimator gain together with a linear feedback gain should be optimized in a robust way.

2.3 Optimal control of a crane

To illustrate the applicability and potential of the robust counterpart formulation (2.12), we consider a crane, which should carry a mass m from a given point to a given target region (cf. Figure 2.1): If the crane's cable with length L is very long with respect to the horizontal excitation $L\sin(\phi)$ of the mass, which is affected by an unknown force F, it is an easy exercise to show that the dynamics of the angle of the line of the crane can be described by the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \phi(t) \\ \dot{\phi}(t) \end{pmatrix} = A(t)\begin{pmatrix} \phi(t) \\ \dot{\phi}(t) \end{pmatrix} + B(t)F(t) + r(t) \tag{2.13}$$

with

$$A(t) := \begin{pmatrix} 0 & 1 \\ -\frac{g}{L(t)} & -\left(b + 2\frac{\dot{L}(t)}{L(t)}\right) \end{pmatrix}, \quad B(t) := \begin{pmatrix} 0 \\ \frac{1}{mL(t)} \end{pmatrix}$$

$$\text{and} \quad r(t) := \begin{pmatrix} 0 \\ -\frac{\ddot{x}}{L} - \frac{\dot{L}\dot{x}}{L^2} \end{pmatrix}$$

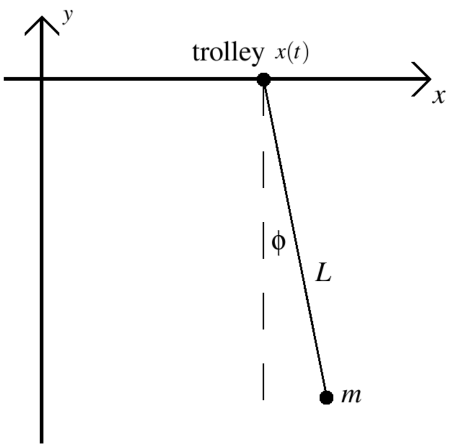

Figure 2.1: A sketch of the crane.

Here, the crane is only considered in a plane $\mathbb{R}^2$, in which the mounting point of the cable is at the time t located at the position $(x(t),0)^T \in \mathbb{R}^2$ while the mass has the position $(x(t)+L(t)\sin(\phi(t)),$ $-L(t)\cos(\phi(t)))^T \in \mathbb{R}^2$. In this notation, $g = 9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}$ is the gravitational constant and $b = 0.1\,\frac{1}{\mathrm{s}}$ a friction coefficient. Note that the above model is only valid for small excitations ϕ where the dynamics can be linearized in the states ϕ and $\dot{\phi}$.

The external force F, acting at the mass in horizontal direction, is assumed to be unknown in our example. The optimal control problem we would like to solve now, assumes that we have the control $u := \left(\ddot{x}, \ddot{L}\right)$ as a degree of freedom to bring the mass as fast as possible from a given point into a desired target region. More precisely, we define the feasible behaviour

$$v := \left(z^T, u^T\right)^T := \left(x, L, \dot{x}, \dot{L}, u^T\right)^T \in \mathbb{V}$$

of the dynamic system by

$$\mathbb{V} := \left\{ v : [0,T] \to \mathbb{R}^{n_6} \;\middle|\; \begin{array}{lcl} \forall t \in [0,T]: & & \\ \dot{z}(t) & = & (\dot{x}(t), \dot{L}(t), u(t)^T)^T \\ z(0) & = & z_0 \\ z(T) & = & z_T \\ v_{\min} & \geq v(t) \geq & v_{\max} \end{array} \right\},$$

where we use the following values for our example

$$\begin{aligned} z_0 &:= \left(0\,\mathrm{m},\ 100\,\mathrm{m},\ 0\,\frac{\mathrm{m}}{\mathrm{s}},\ 0\,\frac{\mathrm{m}}{\mathrm{s}}\right)^T \\ z_T &:= \left(40\,\mathrm{m},\ 100\,\mathrm{m},\ 0\,\frac{\mathrm{m}}{\mathrm{s}},\ 0\,\frac{\mathrm{m}}{\mathrm{s}}\right)^T \\ v_{\min} &:= \left(-10\,\mathrm{m},\ 50\,\mathrm{m},\ -20\,\frac{\mathrm{m}}{\mathrm{s}},\ -20\,\frac{\mathrm{m}}{\mathrm{s}}, -0.3\frac{\mathrm{m}}{\mathrm{s}^2}, -1\frac{\mathrm{m}}{\mathrm{s}^2}\right)^T \\ v_{\max} &:= \left(50\,\mathrm{m},\ 102\,\mathrm{m},\ 20\,\frac{\mathrm{m}}{\mathrm{s}},\ 20\,\frac{\mathrm{m}}{\mathrm{s}}, 0.3\frac{\mathrm{m}}{\mathrm{s}^2}, 1\frac{\mathrm{m}}{\mathrm{s}^2}\right)^T . \end{aligned}$$

We are first interested in the following minimum time optimal control problem for the case that we have no disturbances, i.e. for $F = 0$:

$$\boxed{\begin{array}{ll} \min\limits_{x(\cdot),v(\cdot),p,T} & T \\ \text{s.t.} & \\ \forall t \in [0,T]: & \frac{\mathrm{d}}{\mathrm{d}t}x(t) = A(v(t))x(t) + r(v(t)) \\ \forall t \in [0,T]: & \phi_{\min} \leq \phi(t) \leq \phi_{\max} \\ & x(0) = x_0 \\ & \underline{x}_{\text{target}} \leq x(T) \leq \overline{x}_{\text{target}} \\ & v \in \mathbb{V} \end{array}} \quad , \tag{2.14}$$

where we chose the following numerical values:

$$\begin{aligned} x_0 \quad &:= \quad \left(0\,\text{rad},\ 0\,\frac{\text{rad}}{\text{s}}\right)^T \\ \underline{x}_{\text{target}} \quad &:= \quad \left(-0.042\,\text{rad},\ -0.013\,\frac{\text{rad}}{\text{s}}\right)^T \\ \overline{x}_{\text{target}} \quad &:= \quad \left(0.042\,\text{rad},\ 0.013\,\frac{\text{rad}}{\text{s}}\right)^T \\ \phi_{\max} \quad &:= \quad -\phi_{\min} \ := \ 0.05\,\text{rad}\ . \end{aligned} \tag{2.15}$$

Now, we use the dynamic optimal control package ACADO [1] to solve the above problem by using a piecewise constant control parametrization in combination with an SQP method. The corresponding locally optimal solution, that was obtained with this method, is shown in Figure 2.2. Note that the optimal result for the angle ϕ, the position x, and the cable length L are shown. The path constraints for ϕ are not active in this optimal solution but note that the target constraint $\phi(T) \leq 0.042\,\text{rad}$ is active.

The corresponding value for the minimum time is

$$T = 23.33\text{s}\ . \tag{2.16}$$

Note that it is optimal to reduce the cable length during the movement of the crane.

2.4 Robust optimal control of a crane

In this section we discuss the solution of the robust counterpart optimal control problem which is associated with the problem (2.14). For this aim, we choose $m = 600\,\text{kg}$ and assume that the uncertain force F is bounded by

$$\|F(\cdot)\|_{L_2}^2 \ \leq \ 1200\text{N}^2\text{s}\ . \tag{2.17}$$

We use the optimization software ACADO again to solve also the robustified optimal control problem taking the Lyapunov equation for the state variances into account. Here, we assume that all

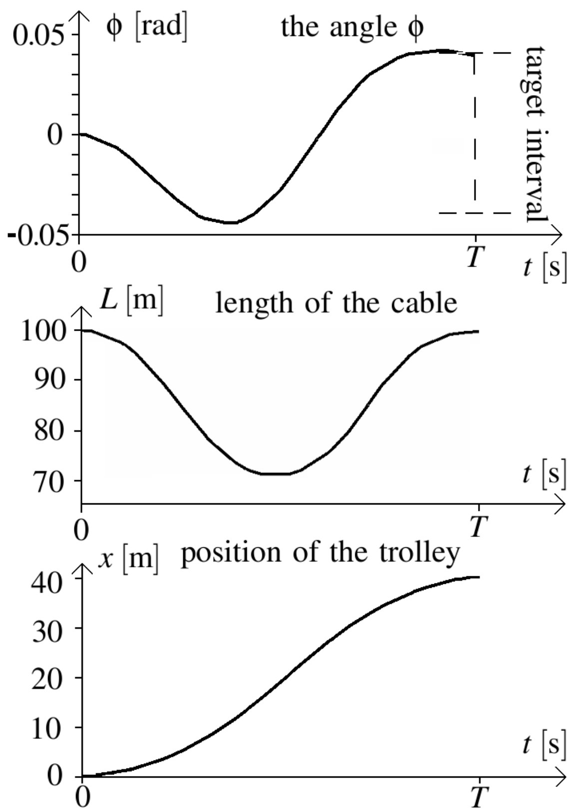

Figure 2.2: A locally optimal result of the optimal control problem (2.14).

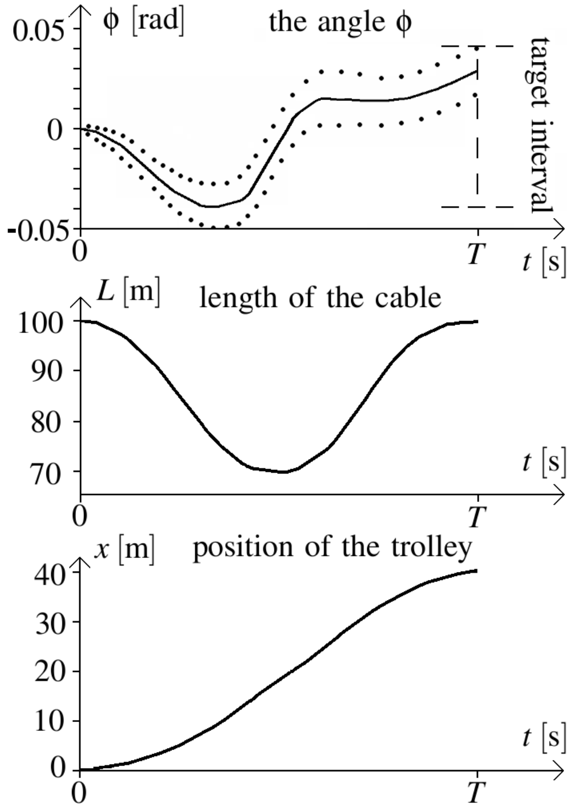

Figure 2.3: A locally optimal result for the robust counterpart problem (2.12) associated with the optimal control problem (2.14).

states are at the time $t = 0$ exactly known, while the path constraints on ϕ as well as the target region constraints should be satisfied for all forces F that are bounded by (2.12). A corresponding robust and locally optimal solution is shown in Figure 2.3. The optimal value for the time T is now larger:

$$T = 24.56\text{s} \tag{2.18}$$

as we need to satisfy more conservative constraints. To visualize that the robustified constraints were active at the optimal solution, which is shown in Figure 2.3, the functions

$$\overline{\phi} := \phi(t) + \gamma\sqrt{\text{Var}(\phi(t))} \text{ and } \underline{\phi} := \phi(t) - \gamma\sqrt{\text{Var}(\phi(t))} \tag{2.19}$$

are plotted as dotted lines. Note that the lower bound of the form

$$\phi(t) - \sqrt{P_{\phi,\phi}(t)} \geq -0.05\,\text{rad}$$

is at active at a certain time $t \in \mathbb{T}$ while the target constraint

$$\phi(T) + \sqrt{P_{\phi,\phi}(T)} \leq 0.042\,\text{rad}$$

is also active.

We have presented methods to design and optimize the stability and robustness of nonlinear dynamic systems with affine uncertainties taking inequality state constraints into account. After reviewing existing concepts for the robustification of linear systems we concentrated on an extension of Lyapunov differential equations for systems with state constraints. We have summarized the interpretation of such constrained Lyapunov differential equations in Theorem 2.2.1 together with the associated remarks showing the equivalence between

- the existence of a feasible solution of the constrained Lyapunov differential equation,
- the exact robustness of the underlying constrained linear dynamic system with respect to L_2-bounded disturbances,
- and the stochastic single chance constraint interpretation for the case of a white noise disturbance.

Furthermore, we transferred the constrained Lyapunov differential equations for special optimal control problems that are linear in the state and the uncertainty while the remaining behaviour can possibly enter in a nonlinear way. In order to demonstrate the applicability of the presented results, we have tested our method for a simple crane model. After a discussion of a time optimal trajectory for this crane we have presented a robustified solution taking inequality state constraints into account.

Chapter 3

Robust design of linear control laws for constrained nonlinear dynamic systems

In this chapter we propose a computationally tractable way of solving robust nonlinear optimal control design problems for time varying uncertainties in a conservative approximation. For this aim, we need to assume that an explicit estimate of the nonlinear terms in the right-hand side function f is given. We demonstrate for a tutorial problem how such an explicit estimate can be constructed illustrating that the results in this discussion are not only of theoretical nature but can also be applied in practice. The results we include from [10] are given without proofs.

3.1 Robust nonlinear optimal control problems

In this section we introduce uncertain optimal control problems for dynamic systems of the form

$$\dot{x}(t) = F(x(t), u(t), w(t)) \ , \ x(0) = 0 \ ,$$

where $x : [0,T] \to \mathbb{R}^{n_x}$ denotes the states, $u : [0,T] \to \mathbb{R}^{n_u}$ the control inputs, and $w : [0,T] \to \mathbb{R}^{n_w}$ an unknown time-varying input which can influence the nonlinear right-hand side function $F : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_x}$. Throughout this chapter, we assume that our only knowledge about the uncertainty w is that it is contained in an uncertainty set Ω_∞ which is defined as

$$\Omega_\infty \ := \ \{ \, w(\cdot) \mid \text{for all } \tau \in [0,T] : \ \| w(\tau) \|_\infty \ \leq \ 1 \ \} \ .$$

In words, Ω_∞ contains the uncertainties $w(\cdot)$ whose L-infinity norm is bounded by 1.

In this chapter, we are interested in designing a feedback law in order to compensate the uncertainties w. Here, we constraint ourselves to the case that the feedback law is linear, i.e. we set $u(t) := K(t)x(t)$ with $K : [0,T] \to \mathbb{R}^{n_u \times n_x}$ denoting the feedback gain. Now, the dynamics of the closed loop system can be summarized as

$$\dot{x}(t) \ = \ f(x(t), K(t), w(t)) \ := \ F(x(t), K(t)x(t), w(t)) \ .$$

Moreover, we assume that we have $f(0,K,0) = 0$ for all $K \in \mathbb{R}^{n_u \times n_x}$, i.e. we assume that $x_{\mathrm{ref}}(t) = 0$ is the steady state which we would like to track. The uncertain optimal gain design problem of our interest can now be stated as

$$\begin{array}{ll} \min\limits_{x(\cdot),K(\cdot)} & \Phi[K(\cdot)] \\ \text{subject to} & \dot{x}(\tau) \;=\; f(x(\tau),K(\tau),w(\tau)) \\ & x(0) \;=\; 0 \\ & C_i(K(\tau))x(\tau) \;\leq\; d_i \quad \text{for all } \tau \in \mathbb{T}_i \end{array} \tag{3.1}$$

with $i \in \{1,\ldots,m\}$. The constraints are assumed to be linear with a given matrix $C : \mathbb{R}^{n_u \times n_x} \to \mathbb{R}^{m \times n_x}$ and a given vector $d \in \mathbb{R}^m$. The sets $\mathbb{T}_i \subseteq [0,T]$ denote the set of times for which the constraints should be satisfied. Here, we can e.g. use $\mathbb{T}_i = [0,T]$ if we want to formulate a path constraint or $\mathbb{T}_i = \{T\}$ if we are interested in a terminal constraint. Note that the above formulation includes the possibility of formulating both state and control bounds as the controls $u(t) = K(t)x(t)$ are linear in x.

Our aim is now to solve the above optimal control problem guaranteeing that the constraints are satisfied for all possible uncertainties $w \in \Omega_\infty$. Thus, we are interested in the following robust counterpart problem:

$$\min_{u(\cdot)} \; \Phi[u(\cdot)] \quad \text{subject to } \; V_i[t,u(\cdot)] \;\leq\; d_i \quad \text{for all } t \in \mathbb{T}_i \; .$$

Here, the robust counterpart functional V is defined component-wise by

$$V_i[t,K(\cdot)] \;:=\; \max_{x(\cdot),w(\cdot)} \; C_i(K(t))x(t) \quad \text{s.t.} \quad \begin{array}{l} \text{for all } \tau \in [0,t]: \\ \dot{x}(\tau) \;=\; f(x(\tau),K(\tau),w(\tau)) \\ x(0) \;=\; 0 \\ w(\cdot) \in \Omega_\infty \; . \end{array} \tag{3.2}$$

Note that the above problem is difficult to solve as it has a bi-level or min-max structure. For the case that f is linear in x and w, the lower-level maximization problem can be regarded as a convex problem as Ω_∞ is a convex set. This lower-level convex case has in a similar context been discussed in [8, 9] where Lyapunov differential equations have been employed in order to reformulate the min-max problem into a standard optimal control problem.

However, for the case that f is nonlinear, the problem is much harder to solve as local maxima in the lower level problem can not be excluded. Our aim is to develop a conservative approximation strategy to over-estimate the functions V_i planning to solve the robust counterpart problem approximately but with guarantees. For this aim, we will have to go one step back within the next section where we start with an analysis of linear dynamic systems. Later, we will come back to a discussion of the more difficult nonlinear problem.

3.2 Linear dynamic systems with time varying uncertainty

In this section, we introduce the basic concept of robust optimization for linear dynamic systems with infinite dimensional uncertainties. We are interested in a dynamic system of the form

$$\dot{x}(t) = A(t)x(t) + B(t)w(t) \quad \text{with} \quad x(0) = 0 \; . \tag{3.3}$$

Here, $x : \mathbb{R} \to \mathbb{R}^{n_x}$ denotes the state while $w : \mathbb{R} \to \mathbb{R}^{n_w}$ is assumed to be a time varying uncertainty. Moreover, $A : \mathbb{R} \to \mathbb{R}^{n_x \times n_x}$ and $B : \mathbb{R} \to \mathbb{R}^{n_x \times n_w}$ are assumed to be given (Lebesgue-) integrable functions.

As outlined in the previous section, we are interested in computing the maximum excitation $V(t)$ of the system at a given time t in a given direction $c \in \mathbb{R}^{n_x}$:

$$V(t) := \max_{x(\cdot),w(\cdot)} \quad c^T x(t) \quad \text{s.t.} \quad \begin{array}{l} \text{for all } \tau \in [0,t]: \\ \dot{x}(\tau) = A(\tau)x(\tau)+B(\tau)w(\tau) \\ x(0) = 0 \\ w(\cdot) \in \Omega_\infty \ . \end{array} \tag{3.4}$$

The above maximization problem can be regarded as an infinite dimensional linear program which is convex as the set Ω_∞ is convex. Following the ideas from [3] we suggest to analyze the dual of the above maximization problem in order to compute V via a minimization problem.

In order to construct the dual problem, we need a time varying multiplier $\lambda : [0,T] \to \mathbb{R}^{n_w}$ to account for the constraints of the form $w_i(\tau)^2 \leq 1$ which have to be satisfied for all times τ and all indices $i \in \{1,\ldots,n_w\}$. Moreover, we express the state function x of the linear dynamic system explicitly as

$$x(t) = \int_0^t H_t(\tau)w(\tau)\,\mathrm{d}\tau \ , \tag{3.5}$$

with the impulse response function $H_t(\cdot) := G(t,\cdot)B(\cdot)$. Here, $G : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{n_x\times n_x}$ denotes the fundamental solution of the linear differential equation (3.3), which is defined as the solution of the following differential equation:

$$\frac{\partial G(t,\tau)}{\partial t} = A(t)G(t,\tau) \quad \text{with} \quad G(\tau,\tau) = 1 \tag{3.6}$$

for all $t,\tau \in \mathbb{R}$.

Now, the dual problem for the function V can be written as

$$\begin{aligned} V(t) &= \inf_{\lambda(\cdot)>0} \max_{w(\cdot)} c^T \left(\int_0^t H_t(\tau)w(\tau)\,\mathrm{d}\tau\right) \\ &\qquad - \sum_{i=1}^{n_w} \int_0^t \lambda_i(\tau)\left(w_i(\tau)^2 - 1\right)\mathrm{d}\tau \\ &= \inf_{\Lambda(\cdot)\succ 0} \int_0^t \frac{c^T H_t(\tau)\Lambda(\tau)^{-1}H_t(\tau)^T c}{4}\,\mathrm{d}\tau \\ &\qquad + \int_0^t \mathrm{Tr}\left[\Lambda(\tau)\right]\mathrm{d}\tau \ . \end{aligned}$$

Here, we use the short hand

$$\Lambda(\tau) := \mathrm{diag}(\lambda(\tau)) \in \mathbb{D}^{n_w}_{++}$$

to denote the diagonal matrix valued function whose entries are the components of the multiplier function λ.

The following Theorem provides a non-relaxed reformulation of the above dual problem such that the associated value function V can be computed more conveniently.

Theorem 3.2.1 *The function V, which is defined to be the optimal value of the optimization problem* (3.4), *can equivalently be expressed as*

$$\begin{array}{rcl} V(t) & = & \inf_{P(\cdot),\theta(\cdot),R(\cdot)\in\mathbb{D}_{++}^{n_w}} \sqrt{1-\theta(\tau)}\sqrt{c^T P(t) c} \\ & & s.t. \left\{ \begin{array}{rcl} \dot{P}(\tau) & = & A(\tau)P(\tau)+P(\tau)A(\tau)^T \\ & & +Tr\left[R(\tau)\right]P(\tau) \\ & & +B(\tau)R^{-1}(\tau)B(\tau)^T \\ P(0) & = & 0 \\ \dot{\theta}(\tau) & = & -Tr\left[R(\tau)\right]\theta(\tau) \\ \theta(0) & = & 1 \end{array} \right. \end{array} \tag{3.7}$$

with $P : [0,T] \to \mathbb{R}^{n_x \times n_x}$ and $\theta : [0,T] \to [0,1]$ being auxiliary states.

The main reason why we are interested in the above theorem is that it allows us to guarantee that the reachable states are independent of the choice of w within an ellipsoidal tube. Let us formulate this result in form of the following corollary:

Corollary 3.2.2 *Let $R : [0,T] \to \mathbb{D}_{++}^{n_w}$ be any given diagonal and positive matrix valued function and $P(t)$ as well as $\theta(t)$ the associated Lyapunov states defined by* (3.7). *If we define the matrix*

$$Q(t) := (1-\theta(t))P(t)$$

as well as the ellipsoidal set

$$\mathscr{E}(Q(t)) \quad := \quad \left\{ Q(t)^{\frac{1}{2}} v \mid v^T v \leq 1 \right\} , \tag{3.8}$$

then we have for all times $t \in [0,T]$ the set inclusion

$$\left\{ \int_0^t H_t(\tau) w(\tau) \mathrm{d}\tau \mid w(\cdot) \in \Omega_\infty \right\} \subseteq \mathscr{E}(Q(t)) .$$

Summarizing the above results, the matrix $Q(t)$ can at each time t be interpreted as the coefficients of an outer ellipsoid $\mathscr{E}(Q(t))$ which contains the set of reachable states at the time t under the assumption that the function w is contained in Ω_∞. In addition, we know from Theorem 3.2.1 that there exists for every direction $c \in \mathbb{R}^{n_x}$ and every time $t \in [0,T]$ a function $R : [0,T] \to \mathbf{cl}(\mathbb{D}_{++}^{n_w})$ such that the associated outer ellipsoid $\mathscr{E}(Q(t))$ touches the set of reachable states in this given direction c at time t.

3.3 A conservative approximation strategy for nonlinear robust optimal control problems

In this section, we come back to the discussion of robust counterpart problems for nonlinear dynamic systems. Here, we are interested in a conservative approximation strategy. Unfortunately, we have to require suitable assumptions on the function f in order to develop such a strategy. We propose to employ the following assumption:

Assumption 3.3.1 *We assume that the right-hand side function f is differentiable and that there exists for each component f_i of the function f an explicit non-linearity estimate $l_i : \mathbb{R}^{n_u \times n_x} \times \mathbb{R}^{n_x \times n_x} \rightarrow \mathbb{R}_+$ with*

$$|f_i(x, K, w) - A_i x - B_i w| \;\leq\; l_i(K, Q) \tag{3.9}$$

for all $x \in \mathscr{E}(Q)$ and for all w with $\|w\|_\infty \leq 1$ as well as all possible choices of K and $Q \succeq 0$. Here, we have used the short hands $A_i := \frac{\partial f_i(0,K,0)}{\partial x}$ and $B_i := \frac{\partial f_i(0,K,0)}{\partial w}$.

From a mathematical point of view, the above assumption does not add a main restriction as we do not even require Lipschitz-continuity of the Jacobian of f. However, in practice, it might of course be hard to find suitable functions l_i which satisfy the above property. Nevertheless, once we find such an upper estimate, tractable conservative reformulations of the original non-convex min-max optimal control problem can be found. This is the aim of this section. In order to motivate how we can find such functions l_i, we consider a simple example:

Example 3.3.2 *Let the function component f_i be convex quadratic in x but linear in w, i.e. we have*

$$|f_i(x, K, w) - A_i x - B_i w| \;=\; x^T S_i(K) x$$

for some positive semi-definite matrix $S_i(K)$. In this case, we can employ the function

$$l_i(K, Q) \;:=\; Tr(\, S_i(K) Q \,)$$

in order to satisfy the above assumption. A less conservative choice would be

$$l_i(K, Q) \;:=\; \lambda_{max}(\, Q^{\frac{1}{2}} S_i(K) Q^{\frac{1}{2}} \,)$$

which would involve a computation of a maximum eigenvalue.

Now, we define the matrix valued function

$$\hat{B} : \mathbb{R}^{n_u \times n_x} \times \mathbb{R}^{n_x \times n_x} \rightarrow \mathbb{R}^{n_x \times n_x} \rightarrow \mathbb{R}^{n_x \times (n_w + n_x)}$$

as

$$\hat{B}(K, Q) \;=\; \left(\frac{\partial f_i(0, K, 0)}{\partial w} \,,\, \mathrm{diag}\,(\, l(K, Q) \,) \right) . \tag{3.10}$$

Theorem 3.3.3 *For any $\hat{R} : [0, T] \rightarrow \mathbb{D}_{++}^{(n_w + n_x) \times (n_w + n_x)}$ and any $K(\cdot)$ regard the solution of the differential equation*

$$\begin{aligned}
\dot{P}(\tau) &= A(K(\tau))P(\tau) + P(\tau)A(K(\tau))^T \\
&\quad + Tr\left[\hat{R}(\tau)\right] P(\tau) \\
&\quad + \hat{B}(K(\tau), Q(\tau))\hat{R}^{-1}(\tau)\hat{B}(K(\tau), Q(\tau))^T \\
P(0) &= 0 \\
\dot{\theta}(\tau) &= -Tr\left[\hat{R}(\tau)\right] \theta(\tau) \\
\theta(0) &= 1
\end{aligned}$$

with $Q(\tau) := [1-\theta(\tau)]\, P(\tau)$. Then for all $t \in [0,T]$ we have the conservative upper bound

$$V_i[t, K(\cdot)] \quad \leq \quad \sqrt{C_i(K(t))\, Q(t)\, C_i(K(t))^T} \tag{3.11}$$

on the worst case functionals V_i which have been defined in (3.2). *Here, we use the notation $A(K) := \frac{\partial f_i(0,K,0)}{\partial x}$.*

3.4 A small tutorial example

Let us demonstrate the applicability of the results by formulating a control design problem for a nonlinear inverted pendulum. The dynamic model is given by

$$\dot{x} = F(x,K,w) \quad = \quad \begin{pmatrix} x_2 \\ \frac{g}{L}\sin(x_1) + \frac{u}{L}\cos(x_1) + \frac{w}{mL^2} \end{pmatrix} . \tag{3.12}$$

Here, g is the gravitational constant while m is the mass, L the length, and x_1 the excitation angle of the pendulum. Note that $\dot{x}_1 = x_2$ is denoting the associated angular velocity. Moreover, u is the controllable acceleration of the joint of the pendulum which can be moved in horizontal direction. For $x = 0$, $u = 0$ and $w = 0$ the pendulum has an unstable steady state. Thus, we will need a feedback control to stabilize the inverted pendulum at this point. Note that there is an uncertain torque w acting at the pendulum.

The right-hand side function f for the closed loop system takes the form

$$f(x,K,w) = \begin{pmatrix} x_2 \\ \frac{g}{L}\sin(x_1) + \frac{Kx}{L}\cos(x_1) + \frac{w}{mL^2} \end{pmatrix} \tag{3.13}$$

where we employ the linear feedback gain $K \in \mathbb{R}^{1\times 2}$ to be optimized. It is possible to show that the function

$$l(K,Q) \quad = \quad \begin{pmatrix} 0 \\ \frac{g}{L} r_1(Q) + \frac{r_2(Q)}{L}\sqrt{KQK^T} \end{pmatrix} \tag{3.14}$$

with

$$r_1(Q) := \left| \sqrt{Q_{1,1}} - \sin(\sqrt{Q_{1,1}}) \right|$$

and

$$r_2(Q) := \left| 1 - \cos(\sqrt{Q_{1,1}}) \right|$$

is an upper bound function satisfying the condition (3.9) within Assumption 3.3.1 for all $K \in \mathbb{R}^{1\times 2}$ and all $Q \in \mathbb{R}^{2\times 2}$ with $\sqrt{Q_{11}} \leq \frac{\pi}{2}$. Note that the above upper estimate l is locally quite tight in the sense that we have at least

$$l(u,Q) \ \leq \ \mathbf{O}(\ \|Q\|^{\frac{3}{2}}\ ) .$$

However, there are also other estimates possible. In the following, we assume that the uncertain torque satisfies $w \in \Omega_\infty$. We are interested in minimizing the L_2 norm of the feedback and estimator gains, i.e. $\int_0^T \|K(t)\|_F^2 \, \mathrm{d}t$, while guaranteeing that path constraints of the form

$$-d \leq x_1(t) \leq d$$

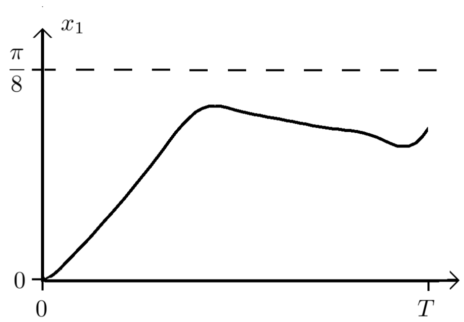

Figure 3.1: A closed-loop simulation of the state x_1 for the torque $w(t) = 1\,\text{Nm}$. The dotted line at $d = \frac{\pi}{8}$ is a conservative upper bound on the worst-case excitation of x_1.

are satisfied in closed loop mode for all possible uncertainties $w \in \Omega_\infty$ and for all times $t \in [0,T]$.

Using Theorem 3.3.3 we can formulate this gain design problem as

$$\inf_{P(\cdot),Q(\cdot),\theta(\cdot),K(\cdot),\hat{R}(\cdot)\in\mathbb{D}^3_{++}} \int_0^T \|K(\tau)\|_F^2 \,\mathrm{d}\tau$$

$$\text{s.t.} \begin{cases} & & \text{for all } \tau \in [0,T]: \\ d & \geq & \sqrt{Q_{11}(\tau)} \\ \dot{P}(\tau) & = & A(K(\tau))P(\tau) + P(\tau)A(K(\tau))^T \\ & & +\mathrm{Tr}\left[\hat{R}(\tau)\right] P(\tau) \\ & & +\hat{B}(K(\tau),Q(\tau))\hat{R}^{-1}(\tau)\hat{B}(K(\tau),Q(\tau))^T \\ Q(\tau) & = & P(\tau)\left[1-\theta(\tau)\right] \\ P(0) & = & 0 \\ \dot{\theta}(\tau) & = & -\mathrm{Tr}\left[\hat{R}(\tau)\right]\,\theta(\tau) \\ \theta(0) & = & 1\,. \end{cases}$$

Note that the above optimization problem is a standard optimal control problem which can be solved with existing nonlinear optimal control software. Any feasible solution of this problem yields a feedback and an estimator gain which guarantees that the path constraints of the form $-d \leq x_1(t) \leq d$ are robustly satisfied for all possible uncertainties $w \in \Omega_\infty$ when running the nonlinear system in closed loop mode. Note that control bounds of the form $-\bar{v} \leq v \leq \bar{v}$ could be imposed in an analogous way as v is linear in x.

In this chapter, the software `ACADO Toolkit` (c.f. [11]) has been employed in order to solve the above optimal control problem with

$$L = 1\,\text{m}\,,\; m = 1\,\text{kg}\,,\; g = 9,81\,\frac{\text{m}}{2^2}\,,$$

$$T = 5\,\text{s}\,,\text{ and }\; d = \frac{\pi}{8}\,.$$

Figure 3.1 shows the state x_1 in a worst-case simulation of the closed-loop system using the optimized feedback gain K. Here, the worst case uncertainty $w(t) = 1\,\text{Nm}$ has been found by local maximization. It is guaranteed that x_1 satisfies the constraints of the form $-d \leq x_1(t) \leq d$ independent of the choice of w but this theoretical result does not state how conservative the result might be. However, the constant uncertainty $w(t) = 1\,\text{Nm}$ turns out to be a local maximizer of x_1 for which

$$\max_{t\in[0,5]} x_1(t) \approx 0.33 \geq \frac{1}{1.19}\frac{\pi}{8}$$

is satisfied. Thus, we can state that in this application the level of conservatism was less than 19%.

Chapter 4

A robustly feasible management problem

In this chapter, we consider a robust feasibility problem for the design of a reference governor: Assume that on the lower level of the two-layer hierarchical control (see Fig. 4.1), a linear feedback controller has already been designed to stabilize the system and to guarantee the output of the linear plant z to track the reference r. On the upper level, a reference governor has to be designed to provide a feasible reference r such that for all disturbances $d \in \mathscr{D}$, the output z belongs to the safe set $\mathscr{Z}$; where $\mathscr{D}$ represents the set of possible disturbances while $\mathscr{Z}$ represents the safety constraints for the operation of the system. Three cases are discussed: 1) the reference is fixed; 2) feedforward management; and 3) affine feedback management. We will show that the resulting necessary and sufficient conditions are affine in the control variables and can be checked in a computationally efficient manner using standard Linear Programming (LP) solvers. Hence, the results can be implemented in supervisory control in SCADA networks, where the computation load is always a problem being focused on.

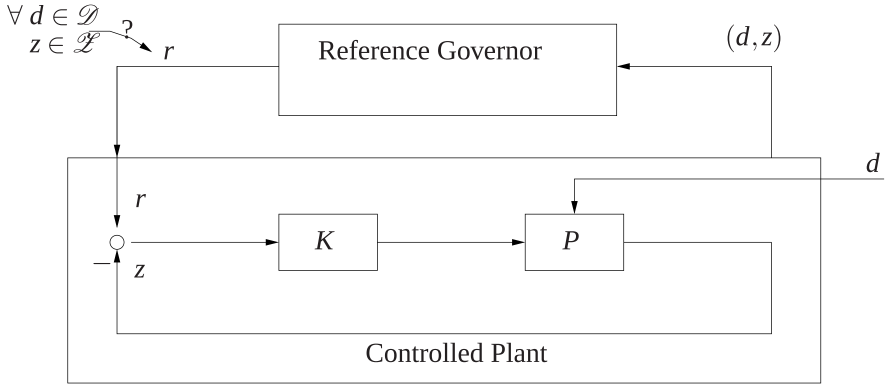

Figure 4.1: Hierarchical control configuration

4.1 Linear state-space model

For clarity, in the remainder of this chapter, the description of the research problem is based on the control of open water networks (which can be seen as a large-scale system composed of many interconnected pools). The discrete lower-level controlled plant is represented in the state-space form as

$$x(k+1) = Ax(k) + Br(k) + Gd(k), \tag{4.1}$$

with the set-point deviation $r(k)$ and the disturbance perturbation $d(k)$. The observation equation is

$$z(k) = Cx(k), \tag{4.2}$$

with the water-level deviation $z(k)$. We define the control and disturbance vectors up to (and excluding) time k by $\mathbf{r} := (r(0), \ldots, r(k-1))^T$ and $\mathbf{d} := (d(0), \ldots, d(k-1))^T$. Note that for a system composed of N subsystems, $r(l) := (r_1(l), \ldots, r_N(l))^T$, and $d(l) := (d_1(l), \ldots, d_N(l))^T$ for $l = 0, \ldots, k-1$; where the water-level set-point and the disturbance in pool i is denoted by r_i and d_i, respectively.

Assume that the system is initially at steady-state, which is $x(0) = 0$, and hence $z(0) = 0$. Up to time k, the system output can be expressed as

$$\mathbf{z} = \mathbf{Br} + \mathbf{Gd}, \tag{4.3}$$

where $\mathbf{z} := (z(1), \ldots, z(k))^T$ with $z(l) := (z_1(l), \ldots, z_N(l))^T$, $\mathbf{B}$ and $\mathbf{G}$ being lower-triangular, Toeplitz matrices with the l-th row given by $\left(CA^{l-1}B, \ldots, CB, 0, \ldots, 0\right)$ and $\left(CA^{l-1}G, \ldots, CG, 0, \ldots, 0\right)$ respectively.

Remark 1 *Most water-level set-points in the practical channel control are calculated from historic data, which can be seen as the nominal set-points to be filtered in Bemporad's reference governor construction [2]. In our discussion, we omitted the nominal set-points in Fig. 4.1 and define $r(k)$ as the set-point deviation at time k. Hence the assumption of the system initial state, i.e. $x(0) = 0$, is reasonable.*

4.2 Admissible sets and the robust feasibility problem

We consider the situation in which the reference vector $\mathbf{r}$ and the disturbance vector $\mathbf{d}$ in (4.3) are unknown but bounded, i.e. we require $\mathbf{r} \in \mathscr{R}$ and $\mathbf{d} \in \mathscr{D}$, where $\mathscr{R}$ and $\mathscr{D}$ are known, bounded sets that define the set of admissible management and disturbance trajectories up to time k.

Remark 2 *In the control of open water channels* $\mathbf{d}$ *contains water demands from farmers. Although these demands are normally scheduled, there exists uncertainty in these disturbances (e.g. starting and stopping time of the water off-takes or the flow needed). This motivates the requirement of* $\mathbf{d} \in \mathscr{D}$*, where* $\mathscr{D}$ *defines the largest water-demand deviation at the downstream ends of pools. The definition of such a set is based on historic data and the environmental consideration, e.g. weather forecasts. Similarly, the requirement of* $\mathbf{r} \in \mathscr{R}$ *is motivated by admissible water-levels in the pools (corresponding to 1) water capacity to satisfy water demands, and 2) channel safety, e.g. no water spillage over the banks of the channel).*

Here we describe the set of admissible reference trajectories by a polytopic model:

$$\begin{aligned} \mathscr{R} &:= \{\mathbf{r} : \|\mathbf{r}_i\|_\infty \leq \sigma_i\} \\ &= \left\{ \mathbf{r} : \begin{bmatrix} I_{k\times k} \\ -I_{k\times k} \end{bmatrix} \begin{bmatrix} r_i(0) \\ \vdots \\ r_i(k-1) \end{bmatrix} \leq \begin{bmatrix} \sigma_i \\ \vdots \\ \sigma_i \end{bmatrix}_{(2k\times 1)} \right\} && (4.4) \\ &= \{\mathbf{r} : \mathbf{R}\Pi\mathbf{r} \leq \sigma\}, && (4.5) \end{aligned}$$

where $\mathbf{R} = \text{diag}\{R,\ldots,R\}$ with $R = [I, -I]^T$ (I being the $kN \times kN$ identity matrix), $\sigma = \left[\sigma_1^T,\ldots,\sigma_N^T\right]^T$ with $\sigma_i \in \mathbb{R}_+^{2k\times 1}$ for $i = 1,\ldots,N$, $\sigma_i = [\sigma_i,\ldots,\sigma_i]^T$. Moreover, $\Pi \in \{0,1\}^{(k\times N)\times(k\times N)}$ is a mapping, stacking the variables in an appropriate way:

$$\Pi = \left[\begin{array}{ccc}
1\ 0\ 0\ \ldots\ 0 & & \\
 & \ddots & \\
 & & 1\ 0\ 0\ \ldots\ 0 \\
\hline
0\ 1\ 0\ \ldots\ 0 & & \\
 & \ddots & \\
 & & 0\ 1\ 0\ \ldots\ 0 \\
\hline
 & \vdots & \\
\hline
0\ 0\ \ldots\ 0\ 1 & & \\
 & \ddots & \\
 & & 0\ 0\ \ldots\ 0\ 1
\end{array}\right].$$

Similarly, the set of admissible disturbances is modeled by

$$\mathscr{D} \quad := \quad \{\mathbf{d} : \mathbf{D}\Pi\mathbf{d} \le \pi\}, \tag{4.6}$$

where $\mathbf{D} = \text{diag}\{D,\ldots,D\}$ with $D = [I,-I]^T$, $\pi = \left[\pi_1^T,\ldots,\pi_N^T\right]^T$ with $\pi_i \in \mathbb{R}_+^{2k\times 1}$ for $i = 1,\ldots,N$, $\pi_i = [\pi_i,\ldots,\pi_i]^T$.

Define the output feasible set $\mathscr{Z}$ as the set of all admissible output trajectories up to time k:

$$\mathscr{Z} \quad := \quad \{\mathbf{z} : \mathbf{Z}\Pi\mathbf{z} \le \tau\}, \tag{4.7}$$

where $\mathbf{Z} = \text{diag}\{Z,\ldots,Z\}$ with $Z = [I,-I]^T$, $\tau = \left[\tau_1^T,\ldots,\tau_N^T\right]^T$ with $\tau_i \in \mathbb{R}_+^{2k\times 1}$ for $i = 1,\ldots,N$, $\tau_i = [\tau_i,\ldots,\tau_i]^T$.

Definition 4.2.1 *(Robustly feasible management) An admissible reference* $\mathbf{r} \in \mathscr{R}$ *is robustly feasible* if and only if *for every admissible disturbance trajectory* $\mathbf{d} \in \mathscr{D}$ *the output trajectory of system (4.3) remains admissible, that is* $\mathbf{z} \in \mathscr{Z}$.

Correspondingly, we have the following formulation of the robust feasibility problem.

Problem 4.2.2 *(Robust feasibility problem) Find necessary and sufficient conditions for the existence of a robust feasible reference trajectory for system (4.3) with sets of admissible reference, disturbance and output trajectories defined by (4.5), (4.6), (4.7) respectively.*

Therefore, the problem is to find if there exists $\mathbf{r} \in \mathscr{R}$ such that for all $\mathbf{d} \in \mathscr{D}$, $\mathbf{z} = \mathbf{B}\mathbf{r} + \mathbf{G}\mathbf{d} \in \mathscr{Z}$.

4.3 Solutions to the robust feasibility problem

We first present the following lemma as the basis for the results in this section.

Lemma 4.3.1 *Given a vector* $\mathbf{v}$ *and a scalar* δ*, the condition* $\mathbf{v}^T\mathbf{d} \le \delta$ *for every* $\mathbf{d} \in \mathscr{D}$ *is satisfied if and only if there exists a* λ *such that the following conditions are satisfied:*

$$\begin{aligned}
&\lambda \ge 0, \\
&(\mathbf{D}\Pi)^T\lambda = \mathbf{v}, \\
&\lambda^T\pi \le \delta.
\end{aligned}$$

Next, we consider three cases for the robust feasibility problem.

4.3.1 The case when the reference is fixed

For the case when the reference in the lower-level system is fixed, i.e. no deviation of the reference, set $\mathbf{r} = 0$ in (4.3). Then Problem 4.2.2 reduces to checking whether

$$\forall \mathbf{d} \in \mathscr{D},\ \mathbf{z} = \mathbf{G}\mathbf{d} \in \mathscr{Z} \tag{4.8}$$

By applying Lemma 4.3.1 row-wise to condition $\mathbf{Z}\Pi\mathbf{z} = \mathbf{Z}\Pi\mathbf{G}\mathbf{d} \leq \tau$, it follows that

Corollary 4.3.2 *Condition (4.8) holds if and only if there exists* $\mathbf{M} = (M_{ij})$ *such that*

$$M_{ij} \geq 0, \tag{4.9}$$

$$(\mathbf{D}\Pi)^T \mathbf{M} = \mathbf{G}^T (\mathbf{Z}\Pi)^T, \tag{4.10}$$

$$\mathbf{M}^T \pi \leq \tau. \tag{4.11}$$

4.3.2 Feedforward management

We then check for the case of feedforward management: Let $\mathbf{r}$ be variable and the robust feasibility problem is to find $\mathbf{r}$ such that

$$\mathbf{r} \in \mathscr{R} \text{ and } \forall \mathbf{d} \in \mathscr{D},\ \mathbf{z} = \mathbf{B}\mathbf{r} + \mathbf{G}\mathbf{d} \in \mathscr{Z} \tag{4.12}$$

Again, applying Lemma 4.3.1 row-wise to condition $\mathbf{Z}\Pi\mathbf{z} = \mathbf{Z}\Pi(\mathbf{B}\mathbf{r} + \mathbf{G}\mathbf{d}) \leq \tau$, it follows that

Corollary 4.3.3 *Condition (4.12) holds if and only if there exists* $\mathbf{M} = (M_{ij})$ *and* $\mathbf{r}$ *such that*

$$M_{ij} \geq 0, \tag{4.13}$$

$$\mathbf{R}\Pi\mathbf{r} \leq \sigma, \tag{4.14}$$

$$(\mathbf{D}\Pi)^T \mathbf{M} = \mathbf{G}^T (\mathbf{Z}\Pi)^T, \tag{4.15}$$

$$\mathbf{M}^T \pi + \mathbf{Z}\Pi\mathbf{B}\mathbf{r} \leq \tau. \tag{4.16}$$

Note the conditions (4.9)-(4.11) for the case of fixed-reference and the conditions (4.13)-(4.16) for the case of feedforward management can be checked using Linear Programming.

4.3.3 Affine feedback management

To consider the case of affine feedback management, we assume the disturbance trajectory $\mathbf{d}$ is measured and the management trajectory $\mathbf{r}$ is an affine function of $\mathbf{d}$. In particular,

$$\mathbf{r}(\mathbf{d}) = \mathbf{w} + \mathbf{L}\mathbf{d}. \tag{4.17}$$

In order to impose that the reference is an affine function of *past* disturbances, we impose $\mathbf{L}$ to be a block lower-triangular matrix.

Remark 3 *This type of control parametrizations has been used in robust MPC (see [7]). In [7], the authors show that control parameterization (4.17) is equivalent to the one where the control is affine function of past states.*

In this case, the robust feasibility problem is to find $\mathbf{w}$ and a block lower-triangular $\mathbf{L}$ such that

$$\forall \mathbf{d} \in \mathscr{D},\ \mathbf{w} + \mathbf{L}\mathbf{d} \in \mathscr{R},\ \mathbf{z} = \mathbf{B}(\mathbf{w} + \mathbf{L}\mathbf{d}) + \mathbf{G}\mathbf{d} \in \mathscr{Z}. \tag{4.18}$$

This problem can be investigated in two steps:

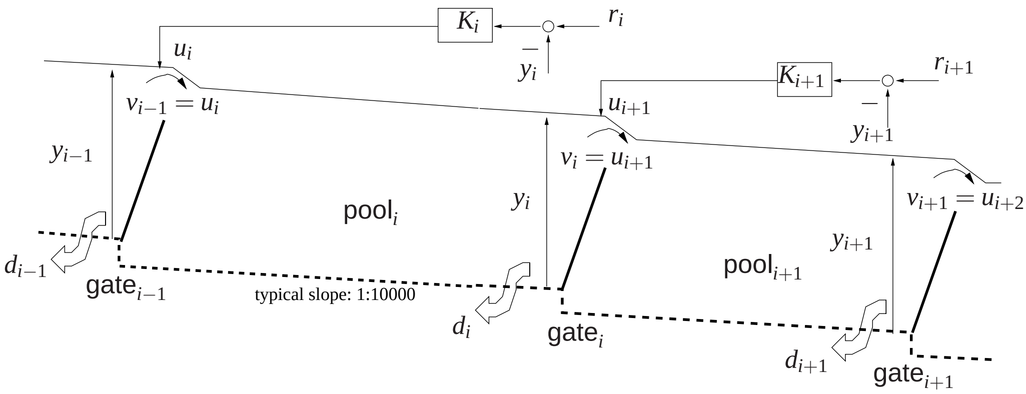

Figure 4.2: Decentralized control of an open water channel

1. The admissibility of the function $\mathbf{r}(\mathbf{d}) = \mathbf{w} + \mathbf{L}\mathbf{d} \in \mathscr{R}$ (L block lower-triangular) for every $\mathbf{d} \in \mathscr{D}$. According to the previous development, this is guaranteed by the following necessary and sufficient condition:

   $\exists$ $\mathbf{N} = (N_{ij})$, $\exists$ $\mathbf{w}$, $\exists$ $\mathbf{L}$ block lower-triangular, s.t.

$$N_{ij} \geq 0, \tag{4.19}$$

$$(\mathbf{D}\Pi)^T \mathbf{N} = \mathbf{L}^T (\mathbf{R}\Pi)^T, \tag{4.20}$$

$$\mathbf{N}^T \pi + \mathbf{R}\Pi\mathbf{w} \leq \sigma. \tag{4.21}$$

   The proof of the above *iff* condition follows the same lines as the proof of Corollary 4.3.3.

2. For fixed $\mathbf{w}$ and $\mathbf{L}$, the admissibility of output $\mathbf{z}$ is such that $\mathbf{B}(\mathbf{w} + \mathbf{L}\mathbf{d}) + \mathbf{G}\mathbf{d} \in \mathscr{Z}$ for every $\mathbf{d} \in \mathscr{D}$. Such a constraint is guaranteed by the following necessary and sufficient condition: $\exists$ $\mathbf{M} = (M_{ij})$ s.t.

$$M_{ij} \geq 0, \tag{4.22}$$

$$(\mathbf{D}\Pi)^T \mathbf{M} = (\mathbf{B}\mathbf{L} + \mathbf{G})^T (\mathbf{Z}\Pi)^T, \tag{4.23}$$

$$\mathbf{M}^T \pi + \mathbf{Z}\Pi\mathbf{B}\mathbf{w} \leq \tau. \tag{4.24}$$

   Again, the proof of the above *iff* condition follows the same lines as the proof of Corollary 4.3.3.

Hence, we have the following theorem for the robust feasibility problem.

Theorem 4.3.4 *Condition (4.18) holds if and only if there exists* $\mathbf{M} = (M_{ij})$, $\mathbf{N} = (N_{ij})$, $\mathbf{w}$, *and block lower-triangular* $\mathbf{L}$ *such that conditions (4.19) - (4.24) are satisfied.*

Since the conditions (4.19) - (4.24) are affine inequalities in the decision variables $\mathbf{M}$, $\mathbf{N}$, $\mathbf{w}$, $\mathbf{L}$, they can be checked using linear programming.

4.4 Case studies

In open water channel control, an important control objective is setpoint regulation of the water-levels in the pools, which enables flow demand at the (often gravity-powered) offtake points to be met without over-supplying [18]. When the number of pools to be controlled is large and the gates widely dispersed, it is natural to employ a decentralized control structure, see Fig. 4.2. The flow into pool$_i$, denoted by u_i, equals the flow supplied by the upstream pool, v_{i-1}. Note that u_i is actually the control action taken by controller K_i to regulate the water-level y_i to a relevant setpoint r_i, in the face of disturbances associated with variations of the uncontrolled offtake load d_i.

In practice, channel capacity is limited. Moreover, the time delay for water to travel from the upstream end to the downstream end of the pool limits the closed-loop bandwidth, which dampens the performance. Hence, the starting and ending of offtakes (d_i) induce transients (i.e. the water-level drops and rises from its setpoint). Such a transient response propagates to upstream pools as regulators take corrective actions [14]. Hence, the open water channel management objectives can be expressed in terms of constraints on the water-levels in each pool: upper bounds avoid water spillage over the banks of the channel; and lower bounds ensure a minimal channel capacity to supply water. In robust reference management, the setpoints are adjusted, which ensures that the water-level constraints are satisfied, in the face of transients associated with load changes within certain constraints.

4.4.1 Plant model

Following [15], the evolution of the water-levels in a channel of N pools with decentralized control can be described by the following continuous state-space model:

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{B}r(t) + \tilde{G}d(t)$$
$$y(t) = \tilde{C}x(t),$$

where $\tilde{A} = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_{p_1} & & \\ & \tilde{A}_2 & \tilde{A}_{p_2} & \\ & & \ddots & \ddots \\ & & & \tilde{A}_N \end{bmatrix}$, $\tilde{B} = \operatorname{diag}\left(\tilde{B}_{r_1}, \dots, \tilde{B}_{r_N}\right)$, $\tilde{G} = \operatorname{diag}\left(\tilde{B}_{d_1}, \dots, \tilde{B}_{d_N}\right)$, and $\tilde{C} = \operatorname{diag}\left(\tilde{C}_1, \dots, \tilde{C}_N\right)$ with $\tilde{A}_i = \begin{bmatrix} 0 & c_{\mathrm{in},i} & -c_{\mathrm{in},i} & 0 \\ 0 & \frac{-2}{t_{\mathrm{d},i}} & \frac{4}{t_{\mathrm{d},i}} & 0 \\ \frac{-\kappa_i}{\rho_i} & 0 & 0 & 1 \\ \frac{-\kappa_i(\rho_i-\phi_i)}{\phi_i\rho_i^2} & 0 & 0 & \frac{-1}{\rho_i} \end{bmatrix}$, $\tilde{A}_{p_i} = \begin{bmatrix} -c_{\mathrm{out},i} \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\tilde{B}_{d_i} = \begin{bmatrix} -c_{\mathrm{out},i} \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\tilde{B}_{r_i} = \begin{bmatrix} 0 \\ 0 \\ \frac{\kappa_i}{\rho_i} \\ \frac{\kappa_i(\rho_i-\phi_i)}{\phi_i\rho_i^2} \end{bmatrix}$, $\tilde{C}_i = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$, where $c_{\mathrm{in},i}$ and $c_{\mathrm{out},i}$ are discharge coefficients, functions of the pool surface area and the gate width; and $t_{\mathrm{d},i}$ is the internal time delay that the water takes to travel from the upstream end to the downstream end of a pool;[1] κ_i, ρ_i and ϕ_i are parameters of the decentralized feedback controller K_i, which is a PI compensator with a low-pass filter. Note that the interconnection between neighboring (controlled) pools $v_i = u_{i+1}$ is expressed in the off-diagonal entries of $\tilde{A}$ (i.e. $\tilde{A}_{p_i}$). To build the prediction model, a discrete-time state-space model of the form (4.1-4.2) is employed. This can be obtained by directly converting the continuous model through a zero-order hold. The sampling interval T_s should be small enough to capture the whole relevant dynamics of the system. In the case studies in Section 4.4.2, the sampling time is set to 5 minutes.

4.4.2 Simulation results

The robust reference governing approach is applied to two pools (i.e. Campbells and Schifferlies) of the East Goulburn Main (EGM) Channel, Victoria, Australia. The parameters of controlled pools are given in Table 4.1. The steady-state water-levels of the two pools are 1.5 and 1.56 m, respectively. The

[1] A first-order Padé approximation is used to represent the transportation time delay $t_{\mathrm{d},i}$. This is reasonable in the modeling since the feedback controller K_i involves a low-pass filter such that high-frequency resonance (caused by the time

Pool	$c_{\text{in},i}$	$c_{\text{out},i}$	τ_i
1	0.055	0.036	5 min
2	0.017	0.026	6 min
Controller	κ_i	ϕ_i	ρ_i
1	0.74	71.83	8.52
2	1.19	141.27	16.75

Table 4.1: Parameters of (controlled) pools

Case	Maximum disturbance deviation		Maximum reference deviation		Maximum water-level deviation	
	π_1	π_2	σ_1	σ_2	τ_1	τ_2
	(Ml/day)		(m)		(m)	
Fixed reference	7	6	0	0	0.10	0.06
Feedforward management	11	7	0.08	0.05	0.10	0.06
Feedback management	50	35	0.08	0.05	0.10	0.06

Table 4.2: Parameters describing admissible disturbance, reference and water-level trajectories

prediction horizon is 480 steps (of 5 minutes), which corresponds to a forecast of 40 hours. Following the procedure outlined in Section 4.1, the matrices **B** and **G** in (4.3) are constructed. The polytope $\mathscr{X}$ that models the admissible output trajectories for the two pools is set as: τ_1 and τ_2 are constant vectors with entries 0.1 m and 0.06 m, respectively. These requirements impose the constraint that the water-level deviations must remain within ± 0.1 m (in pool_1) and ± 0.06 m (in pool_2) throughout the time horizon. We solve the robust feasibility problem for the following cases: 1) without reference deviation, 2) feedforward management, and 3) feedback management. So, we check for each of the three cases by equations (4.9-4.11), (4.13-4.16), and (4.19-4.24), respectively, for the existence of a robustly feasible solution.

For the case of feedforward and of disturbance feedback management, the polytope $\mathscr{R}$ that models the admissible reference trajectories for the two pools is set as: σ_1 and σ_2 are constant vectors with entries 0.08 m and 0.05 m, respectively. These requirements impose the constraint that the water-level setpoint deviations must remain within ± 0.08 m (in pool_1) and ± 0.05 m (in pool_2) throughout the time horizon. The polytope $\mathscr{D}$ that models the set of admissible disturbance trajectories is defined by setting π_1 and π_2 as constant vectors with entries π_1 and π_2, respectively. Starting from small π_1 and π_2, the set $\mathscr{D}$ is systematically increased until the conditions for existence of the robustly feasible solution are no longer feasible.[2] For the three cases, the maximum values of π_1 and π_2 for which these conditions remain feasible are listed in Table 4.2. We see that for the case without reference variation and for the case of feedforward management, the admissible set of disturbance trajectories is much smaller than for the case of feedback management, which is within expectation.

In order to test the performance of the feedback management, the disturbance trajectory is set as shown in Fig. 4.3; note that the largest disturbance deviations (in pool_1 and pool_2) correspond to the

delay) is dampened.

[2]The bisection method has been used for the selection of π_1 and π_2. Note that in the simulation, priority was given to π_2, considering the propagation of the system transients in the upstream direction [14].

maximum admissible disturbances listed in Table 4.2. For comparison, the response of the lower-level system with the original references (the thick dash-dotted lines) is also given (see the thin dash-dotted lines in Fig. 4.4). The upper bound and lower bound constraints on the water-levels are violated at some time instants (around 275 min and around 1500 min) in the prediction horizon. In contrast, under the calculated references (the thick solid lines), the dynamics of the system is within the water-level constraints (see thin solid lines in Fig. 4.4).

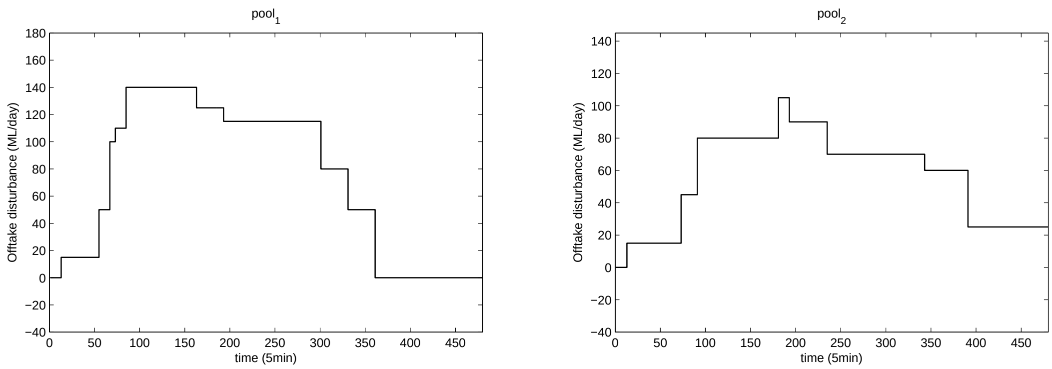

Figure 4.3: Off-take disturbances in $pool_1$ and $pool_2$

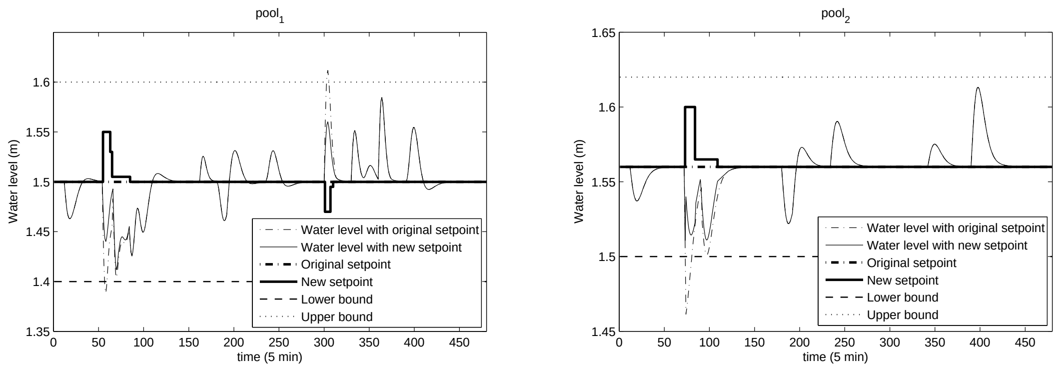

Figure 4.4: Reference governing for feedback case with constraint on off-take demand; $pool_1$ and $pool_2$

This section has discussed the formulation of a robust feasibility problem for the design of reference governors in a two-layer hierarchical control. The constraints on the admissible set of disturbance, reference, and output trajectories are incorporated in the formulation of the robust governor. Necessary and sufficient conditions that are affine in the decision variables are given. Using LP solvers, these conditions can be checked efficiently. The proposed reference governor design approach can be applied in supervisory control in SCADA networks.

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