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Authors:	A. Núñez, B. Picasso, D. De Vito, R. Scattolini, P. Colaneri, J. Espinosa, F. Valencia, B. De Schutter, K. Stankova

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Project co-ordinator

Name: Bart De Schutter
Address: Delft Center for Systems and Control
Delft University of Technology
Mekelweg 2, 2628 Delft, The Netherlands
Phone Number: +31-15-2785113
Fax Number: +31-15-2786679
E-mail: b.deschutter@tudelft.nl

Project web site: `http://www.ict-hd-mpc.eu`

Executive Summary

This report describes the HD-MPC research activities focusing on WP3 – "Development of hierarchical and distributed MPC methods" on more specifically on Task 3.3 — "Coordination mechanisms".
The report is organized in two main chapters:
Chapter 1 discusses an MPC approach to the design of hierarchical control systems. The methodology is presented for the design of two level hierarchical control systems. The higher level corresponds to a system with slow dynamics and whose control inputs must be provided by the subsystems with faster dynamics and placed at the lower level. MPC control laws are synthesized for both the levels and overall convergence properties are established. The use of different control configurations is also considered by allowing the switching on/off of the subsystems at the lower level. A simulation example is reported to witness the potentialities of the proposed solution.
Chapter 2 proposes a feasible cooperation distributed model predictive control scheme, based on game theory. In this chapter a distributed model predictive control scheme is formulated as a decision problem, where the decisions of each subsystem affect not only the decisions of the other subsystems but the whole system performance. The decision problem is of the type "bargaining game". This formulation allows each subsystem to decide whether to cooperate or not depending on the benefits that the subsystem can gain from the cooperation. A game-theoretic based solution is proposed. The proposed control scheme is tested on case studies with a chain of two continuous stirred tank reactors followed by a non-adiabatic flash separator.

Chapter 1

An MPC approach to the design of hierarchical control systems

The contents of this chapter have been developed by Bruno Picasso, Daniele De Vito, Riccardo Scattolini and Patrizio Colaneri (Politecnico di Milano, Dipartimento di Elettronica e Informazione).

1.1 Introduction

Hierarchical control has been receiving great attention within the scientific community for many years, see for example the visionary books [30, 12], the recent survey papers [58, 53] and the many references therein. This interest is motivated by the large number of control problems which can be handled with hierarchical control structures more suitably than with classic control methodologies.
Among them, it is possible to mention the design of controllers for multi time scale systems characterized by clearly separable slow and fast dynamics, see [12, 23, 57], or [3, 61] for a couple of significant industrial examples, the former concerning a waste water treatment plant, while the latter a greenhouse control problem. Under a slightly different standpoint, a set of representations of the plant to be controlled, each characterized by a different degree of abstraction, is considered in [43].
Hierarchical control structures can also be used to coordinate a number of local controllers, such as in [20] where the control of a power plant has been considered, or in [40, 41] dealing with transportation and power networks, respectively. In [2], a decentralized and hierarchical control technique has been applied to interconnected dynamical systems incorporating elements tied to bounded uncertainty.
Another important class of problems, including the one studied in this report, concerns the design of a controller for plants characterized by an inherent hierarchical structure. The highest layer of the hierarchy corresponds to a system with slow dynamics and whose control inputs must be effectively provided by subsystems (actuators) with faster dynamics and placed at lower layers. In these problems, a cascade control structure is typically used: the regulator at the higher layer computes the desired control actions, while local regulators at the lower layers are in charge of controlling the actuators. An independent design of the regulators at different layers is often performed assuming complete frequency decoupling and perfect reference tracking of the inner loops, i.e., the control loops at the lower layers. If this hypothesis can not be taken for granted, some kind of information exchange among layers is required to guarantee both the feasibility of the reference signals computed at the higher level and the overall stability. In any case, a kind of cooperation among the subsystems operating at the same layer is advisable, especially in overactuated plants, often built up for physical

redundancy purposes to tackle damage events or to meet secondary objectives. The problem of distributing the control effort among a number of actuators is usually called control allocation. Technical literature encloses many results dealing with the control allocation problem, warranted, for instance, by fault tolerance and reconfiguration ability requirements [13, 75] along with stability and performance constraints [67, 25], in applications ranging from the automotive [60] to the aerospace [54], aircraft [26, 66], robotics [33, 70, 76], marine [18, 19, 59], power of wireless nodes [1], demands in free market [34] fields, and so on. While early contributions neglected the low level subsystems' dynamical behavior, in more recent papers, dynamical behavior of the actuators has been considered, although the stability issue is still a largely open problem, see e.g., [26].
In this report, a hierarchical control design problem for a two layer cascade control structure as described above, has been investigated. The high level controller operates at a slower time scale than the actuators placed at the low layer. Different control configurations are also considered by allowing the possibility to on-line switch on and off some actuators.
A robust control approach has been undertaken to obtain a convergence result for the overall system. This approach stems from considering the discrepancy between the ideal control actions, requested by the high level controller, and those actually achieved by the actuators as a disturbance term to be rejected in the design phase of the high level controller.
The problem has been tackled by resorting to the Model Predictive Control (MPC) approach. This choice is motivated by the ever increasing popularity of MPC in the process industry, see e.g., the survey papers [48, 49], along with its well established theoretical foundations, see e.g., [29]. Following the ideas preliminarily developed in [52], at the high level a switched robust MPC algorithm is used to design a stabilizing feedback control law and to select the optimal subset of actuators to be activated. At the low level, a number of standard MPC problems are set up at any instant of the fast time scale to solve the corresponding tracking problems.
Differently from the methods previously described, the switched robust MPC approach stated here allows one to simultaneously ensure global stabilization, control configuration selection for performance optimization and possible state and control constraints fulfillment while readily accounting for actuators' dynamics. Furthermore, the proposed approach differs from the classical sequential design, where the lower level control loop is considered either as part of the model seen by the upper level controller or it is supposed to be at steady state. On the contrary, the adoption of robust control means allows one to largely decouple the design at the two levels of the hierarchical structure even if the frequency decoupling principle can not be assumed.

The report is organized as follows: in Chapter 1.2, the systems at the high and low levels are defined and the considered hierarchical control structure is introduced. Chapter 1.3 describes the MPC algorithms used at the two levels and presents the main convergence results related to the overall controlled system. A simulation example is presented in Chapter 1.4 to witness the performance of the proposed approach. Finally, some conclusions are drawn in Chapter 1.5.

Notation. In order to cope with a multi-rate implementation typical of hierarchical control structures, where the upper layer acts at a slower rate than the lower layer, two time scales are considered: the fast discrete-time index is denoted by h, while the slow discrete-time index is represented by k. Then, given a signal $\phi^f(h)$ in the fast time scale, its sampling in the slow time scale is $\phi(k) = \phi^f(\tau k)$, where τ is a fixed positive integer.
I_k is the identity matrix of dimension k, $0_{k,l}$ is the null matrix in $\mathbb{R}^{k\times l}$.
By $\|\cdot\|$ we denote the Euclidean vector or induced matrix norm. For $x \in \mathbb{R}^n$ and $\mathbb{R}^{n\times n} \ni P > 0$, we let $\|x\|_P = \sqrt{x'Px}$.

1.2 Problem formulation

The process to be controlled is described by the discrete-time system

$$\mathscr{P}: \; x^f(h+1) = A^f x^f(h) + \sum_{i=1}^{m} \alpha_i^f(h) b_i^f u_i^f(h), \tag{1.1}$$

where $x^f \in \mathbb{R}^{n_x}$ is the measurable state, $u_i^f \in \mathbb{R}^{n_i}$ is the control variable provided by the i-th actuator ($i = 1, \dots, m$), and $\alpha_i^f(h) \in \{0,1\}$ is an additional input variable such that

$$\alpha_i^f(h) = \begin{cases} 1 & \text{if the } i\text{-th actuator is active at time } h \\ 0 & \text{otherwise}. \end{cases}$$

The control variables u_i^f's coincide with the outputs $\tilde{u}_i$'s of the systems modeling the m actuators. The i-th actuator is described by the following linear dynamical system:

$$\mathscr{S}_{\mathrm{act}\,i}: \; \begin{cases} \zeta_i(h+1) = F_i \zeta_i(h) + G_i \nu_i(h), \quad \zeta_i(0) = \zeta_{i0} \\ \tilde{u}_i(h) = H_i \zeta_i(h), \end{cases} \tag{1.2}$$

where $\zeta_i \in \mathbb{R}^{n_{\varsigma i}}$ is the measurable state and $\nu_i \in \mathbb{R}^{n_{\nu i}}$ is the manipulated input.
The control goal is to design a controller that stabilizes the cascade interconnection of systems (1.1) and (1.2). To this end, a two layer hierarchical controller is proposed. At the upper level, a controller is designed at a slow time scale, which selects a suitable subset of the actuators to be used in the next (long) time interval, namely the values of α_i, and computes the control action u_i the i-th actuator should ideally produce. Such a u_i is considered as the reference signal for system $S_{\mathrm{act}\,i}$. Accordingly, a number of tracking problems are solved by the lower level controllers, which regulate the actuators by operating at a faster time scale. Thus, the effective control actions $\tilde{u}_i$'s are provided to the process. Due to dynamic and intrinsic limitations of the actuators, in general one has $\tilde{u}_i \neq u_i$, so that a robustness issue arises. This problem is handled by modeling the discrepancy $w_i^f = \tilde{u}_i - u_i$ between the ideal control action and the one provided by the i-th actuator as a disturbance to be rejected in the design phase of the high level controller.

In the remaining part of this Chapter, the formal framework needed to work out the described plan is introduced and the suitable assumptions for the considered models are specified.

1.2.1 The process: model for the high level controller

As far as system (1.1) is concerned, the following control constraint is considered: for $i = 1, \dots, m$,

$$u_i^f \in \mathscr{U}_i,$$

$\mathscr{U}_i \subset \mathbb{R}^{n_i}$ being a compact set containing the origin as an interior point. The total number of control variables is $n_u = \sum_{i=1}^{m} n_i$. We let

$$\mathscr{U} = \mathscr{U}_1 \times \cdots \times \mathscr{U}_m \subset \mathbb{R}^{n_u}.$$

Consider

$$B^f = \begin{bmatrix} b_1^f & b_2^f & \cdots & b_m^f \end{bmatrix} \in \mathbb{R}^{n_x \times n_u}, \tag{1.3}$$

it is supposed that

Assumption 1 *The pair (A^f, B^f) is stabilizable.*

In accordance with the description of the high level controller reported at the beginning of this Chapter, for some fixed integer $\tau \geq 1$, let us decompose the control variables u_i^f's of system (1.1) in the form $u_i^f(h) = \bar{u}_i(h) + (u_i^f(h) - \bar{u}_i(h))$, where $\bar{u}_i(h) \in \mathscr{U}_i$ is some piecewise constant signal such that, $\forall k \in \mathbb{N}$ and $\forall j = 0, \ldots, \tau - 1$, it holds that $\bar{u}_i(\tau k + j) = \bar{u}_i(\tau k)$. Then system (1.1) can be rewritten as

$$x^f(h+1) = A^f x^f(h) + \sum_{i=1}^{m} \alpha_i^f(h) b_i^f \bar{u}_i(h) + \sum_{i=1}^{m} \alpha_i^f(h) b_i^f w_i^f(h), \tag{1.4}$$

where $w_i^f(h) = u_i^f(h) - \bar{u}_i(h)$ is considered as a matched disturbance term. Also the signals $\alpha_i^f(h)$ are supposed to be piecewise constant: specifically, $\forall k \in \mathbb{N}$ and $\forall j = 0, \ldots, \tau - 1$, it holds that

$$\alpha_i^f(\tau k + j) = \alpha_i^f(\tau k). \tag{1.5}$$

Letting

$$x(k) = x^f(\tau k), \quad u_i(k) = \bar{u}_i(\tau k), \quad \alpha_i(k) = \alpha_i^f(\tau k)$$

and

$$A = (A^f)^\tau, \quad b_i = \sum_{j=0}^{\tau-1} (A^f)^{\tau-j-1} b_i^f, \quad w_i(k) = \sum_{j=0}^{\tau-1} (A^f)^{\tau-j-1} b_i^f w_i^f(\tau k + j), \tag{1.6}$$

system (1.4) can be written in the slow sampling rate as

$$x(k+1) = Ax(k) + \sum_{i=1}^{m} \alpha_i(k) b_i u_i(k) + \sum_{i=1}^{m} \alpha_i(k) w_i(k). \tag{1.7}$$

It is useful to simplify the notation as follows: let the set of active actuators at time k be identified by one of the 2^m values taken on by the vector

$$\alpha(k) = \begin{bmatrix} \alpha_1(k) \; \alpha_2(k) \; \cdots \; \alpha_m(k) \end{bmatrix} \in \{0,1\}^m.$$

For a given $\alpha \in \{0,1\}^m$, we let $\mathscr{I}(\alpha) = \{i \,|\, \alpha_i = 1\}$ and

$$\begin{aligned} \mathscr{A}_1 &= \mathrm{diag}\{\alpha_1 I_{n_1}, \cdots, \alpha_m I_{n_m}\} \in \mathbb{R}^{n_u \times n_u} \\ \mathscr{A}_2 &= \mathrm{diag}\{\alpha_1 I_{n_x}, \cdots, \alpha_m I_{n_x}\} \in \mathbb{R}^{mn_x \times mn_x} \end{aligned} \tag{1.8}$$

(notice that the dependence of $\mathscr{A}_1$ and $\mathscr{A}_2$ on α is omitted).
We also define

$$u(k) = \begin{bmatrix} u_1(k) \\ \vdots \\ u_m(k) \end{bmatrix} \in \mathscr{U} \subset \mathbb{R}^{n_u}, \qquad w(k) = \begin{bmatrix} w_1(k) \\ \vdots \\ w_m(k) \end{bmatrix} \in \mathbb{R}^{mn_x} \tag{1.9}$$

and

$$B_1 = \begin{bmatrix} b_1 & b_2 & \cdots & b_m \end{bmatrix} \in \mathbb{R}^{n_x \times n_u}, \tag{1.10a}$$
$$B_2 = \begin{bmatrix} I_{n_x} & I_{n_x} & \cdots & I_{n_x} \end{bmatrix} \in \mathbb{R}^{n_x \times mn_x}. \tag{1.10b}$$

Accordingly, system (1.7) can be rewritten as

$$\mathscr{P}_{\mathrm{slow}}: \; x(k+1) = Ax(k) + B_1 \mathscr{A}_1(k) u(k) + B_2 \mathscr{A}_2(k) w(k). \tag{1.11}$$

It is helpful to introduce the following augmented system: let $\chi = \begin{bmatrix} x \\ \mu \end{bmatrix} \in \mathscr{X} = \mathbb{R}^{n_x} \times \mathscr{U}$, define $\mu(k+1) = \mathscr{A}_1(k)u(k)$, then consider

$$\mathscr{P}_{\text{slow}}^{\text{aug}}: \ \chi(k+1) = \mathscr{A}\chi(k) + \mathscr{B}_1\mathscr{A}_1(k)u(k) + \mathscr{B}_2\mathscr{A}_2(k)w(k), \tag{1.12}$$

where

$$\mathscr{A} = \begin{bmatrix} A & 0_{n_x,n_u} \\ 0_{n_u,n_x} & 0_{n_u,n_u} \end{bmatrix}, \quad \mathscr{B}_1 = \begin{bmatrix} B_1 \\ I_{n_u} \end{bmatrix}, \quad \mathscr{B}_2 = \begin{bmatrix} B_2 \\ 0_{n_u,mn_x} \end{bmatrix}.$$

Associated with this system, we also consider the output transformation

$$z(k) = \begin{bmatrix} \chi(k) \\ \mathscr{A}_1(k)u(k) \end{bmatrix} = \begin{bmatrix} x(k) \\ \mathscr{A}_1(k-1)u(k-1) \\ \mathscr{A}_1(k)u(k) \end{bmatrix}. \tag{1.13}$$

To sum up, at the high level, a robust stabilization problem has to be solved for system (1.12) which returns $u(k)$ and $\alpha(k)$ (namely, the piecewise constant signals $\bar{u}_i(h)$ in equation (1.4) and $\alpha_i^f(h)$, $i = 1,\dots,m$).

1.2.2 The actuators: model for the low level controller

As for the actuators, the following state and input constraints are considered:

$$\zeta_i \in \mathscr{Z}_i, \quad \nu_i \in \mathscr{V}_i, \tag{1.14}$$

where $\mathscr{Z}_i$ and $\mathscr{V}_i$ are compact sets containing the origin as an interior point. For systems (1.2), it is further supposed that

Assumption 2 *For all $i = 1,\dots,m$, it holds that:*

1. *The pair (F_i, G_i) is reachable.*
2. *The number of control variables of the i-th actuator is equal to the number of its output variables, i.e., $n_{\nu_i} = n_i$.*
3. *System* (1.2) *has no invariant zeros equal to 1, i.e.,*

$$\text{rank} \begin{bmatrix} I_{n_{\varsigma_i}} - F_i & -G_i \\ H_i & 0_{n_i,n_i} \end{bmatrix} = n_{\zeta_i} + n_i.$$

Moreover, for any $\hat{u}_i \in \mathscr{U}_i$, there exists a pair $(\hat{\zeta}_i, \hat{\nu}_i) \in \mathscr{Z}_i \times \mathscr{V}_i$ such that

$$\begin{bmatrix} I_{n_{\varsigma_i}} - F_i & -G_i \\ H_i & 0_{n_i,n_i} \end{bmatrix} \begin{bmatrix} \hat{\zeta}_i \\ \hat{\nu}_i \end{bmatrix} = \begin{bmatrix} 0_{n_{\varsigma_i},1} \\ \hat{u}_i \end{bmatrix}.$$

Thus,

$$\begin{cases} \hat{\zeta}_i(\hat{u}_i) = \Gamma_i \hat{u}_i \\ \textit{with} \quad \Gamma_i = \begin{bmatrix} I_{n_{\varsigma_i}} & 0_{n_{\varsigma_i},n_i} \end{bmatrix} \begin{bmatrix} I_{n_{\varsigma_i}} - F_i & -G_i \\ H_i & 0_{n_i,n_i} \end{bmatrix}^{-1} \begin{bmatrix} 0_{n_{\varsigma_i},n_i} \\ I_{n_i} \end{bmatrix} \in \mathbb{R}^{n_{\varsigma_i} \times n_i}. \end{cases} \tag{1.15}$$

4. $\{H_i\zeta_i \,|\, \zeta_i \in \mathscr{Z}_i\} = \mathscr{U}_i$.

5. *There exists a positive integer* ℓ_i *such that, from any initial state* $\zeta_{i0} \in \mathscr{Z}_i$*, it is possible to reach any equilibrium state* $\hat{\zeta}_i \in \mathscr{Z}_i$ *satisfying* (1.15) *in* ℓ_i *steps under the state and control constraints* $(\zeta_i(h), \nu_i(h)) \in \mathscr{Z}_i \times \mathscr{V}_i$*,* $\forall h = 0, \ldots, \ell_i - 1$*. Let* $\ell = \max_{i=1,\ldots,m} \ell_i$*.*

Remark 1 *Regarding Assumption 2.3, notice that, the lack of invariant zeros equal to 1 does not imply the existence of an equilibrium pair* $(\hat{\zeta}_i, \hat{\nu}_i) \in \mathscr{Z}_i \times \mathscr{V}_i$ *for any* $\hat{u}_i \in \mathscr{U}_i$ *because of the state and input constraints* (1.14)*. For the same reason, Assumption 2.1 does not imply Assumption 2.5.*
Assumption 2.4 is a consistency hypothesis among the layers. Indeed, on the one hand it guarantees that any reference $u_i(k)$ *provided by the high level controller is feasible for the i-th low level subsystem. On the other hand it ensures that any control action* $\tilde{u}_i(h)$ *performed by the i-th actuator satisfies the control constraint on the process, no matter the state of the actuator is.*

At the low level, a number of tracking problems have to be solved. For a given constant reference $\hat{u}_i \in \mathscr{U}_i$, according to Assumption 2.3, let $(\hat{\zeta}_i, \hat{\nu}_i) \in \mathscr{Z}_i \times \mathscr{V}_i$ be the corresponding equilibrium pair. Then system (1.2) can be rewritten as

$$\begin{cases} \delta\zeta_i(h+1) = F_i\delta\zeta_i(h) + G_i\delta\nu_i(h), \quad \delta\zeta_i(0) = \delta\zeta_{i0} \\ \delta u_i(h) = H_i\delta\zeta_i(h), \end{cases} \tag{1.16}$$

where $\delta\zeta_i(h) = \zeta_i(h) - \hat{\zeta}_i$, $\delta\nu_i(h) = \nu_i(h) - \hat{\nu}_i$ and $\delta u_i(h) = \tilde{u}_i(h) - \hat{u}_i$. Consequently, the state and input constraints (1.14) take the equivalent form

$$\begin{cases} \delta\zeta_i \in \mathscr{Z}_i - \hat{\zeta}_i \;=\; \{\zeta_i - \hat{\zeta}_i \,|\, \zeta_i \in \mathscr{Z}_i\} \\ \delta\nu_i \in \mathscr{V}_i - \hat{\nu}_i \;=\; \{\nu_i - \hat{\nu}_i \,|\, \nu_i \in \mathscr{V}_i\}. \end{cases} \tag{1.17}$$

The state feedback controller to be designed is denoted by

$$\delta\nu_i(h) = \varphi_i(\delta\zeta_i(h), \hat{u}_i) \tag{1.18}$$

where, for later use, we find it convenient to explicitly show the dependence of the control law from the reference $\hat{u}_i$.

1.2.3 The hierarchical controller

In the hierarchical implementation of the controller, at any long sampling instant k, the controller at the upper level provides the vector $\alpha(k)$ identifying the active actuators for the successive τ instants (in the fast time scale) and the value $u(k)$ of the input vector for system (1.12). Then, consistently with the partition (1.9) of such a vector, the components $\alpha_i(k)u_i(k)$ of $\mathscr{A}_1(k)u(k)$ are considered as the references for the lower level controllers of the corresponding systems (1.2). The references are updated at the successive long sampling instant.
More formally, consider the signals $\alpha(k)$ and $u(k)$, $k = 0, 1, \ldots$, generated by the controller at the upper level and, $\forall i = 1, \ldots, m$, define

$$\begin{cases} \bar{u}_i(h) \;=\; u_i\left(\lfloor \frac{h}{\tau} \rfloor\right) \\ \alpha_i^f(h) \;=\; \alpha_i\left(\lfloor \frac{h}{\tau} \rfloor\right), \end{cases} \tag{1.19}$$

where $\lfloor \cdot \rfloor$ is the floor function. Let

$$\hat{u}_i(h) = \alpha_i^f(h)\bar{u}_i(h) \tag{1.20}$$

be the reference for the i-th system (1.2). Hence, in view of (1.18), the control signal for the i-th actuator is

$$\nu_i(h) = \varphi_i(\zeta_i(h) - \hat{\zeta}_i(h), \hat{u}_i(h)) + \hat{\nu}_i(h), \tag{1.21}$$

$(\hat{\zeta}_i(h),\, \hat{\nu}_i(h))$ being the equilibrium pair associated to $\hat{u}_i(h)$.

Remark 2 *Notice that, although the reference $\hat{u}_i(h)$ is a piecewise constant function, the low level controller computes $\nu_i(h)$ by considering the current value $\hat{u}_i(h)$ as a constant reference from the current (fast) time instant onwards.*

If $\alpha_i(k) = 1$, the i-th control variable for system (1.1) is defined by the corresponding output of system (1.2), (1.21), that is

$$u_i^f(h) = \tilde{u}_i(h), \tag{1.22}$$

where

$$\begin{cases} \zeta_i(h+1) = F_i\zeta_i(h) + G_i\nu_i(h) \\ \nu_i(h) = \varphi_i(\zeta_i(h) - \hat{\zeta}_i(h), \hat{u}_i(h)) + \hat{\nu}_i(h) \\ \tilde{u}_i(h) = H_i\zeta_i(h). \end{cases} \tag{1.23}$$

The difference between the control value $u_i(k)$ provided by the controller at the upper level and the one achieved by the i-th subsystem at the lower level is the matched disturbance term appearing in (1.4), namely

$$w_i^f(h) = \tilde{u}_i(h) - \bar{u}_i(h). \tag{1.24}$$

Conversely, if $\alpha_i(k) = 0$, the internal state ζ_i of the actuator is driven to 0 whilst the value of $\tilde{u}_i(h)$ is not relevant, as well as those of $\bar{u}_i(h)$ and $w_i^f(h)$.

1.3 Design of the MPC hierarchical controller

In order to tackle the robust control problem for the synthesis of the high level controller, while coping with the presence of state and control constraints at both the lower and the upper levels, the Model Predictive Control approach is considered.

To have versatility in the definition of the cost functions, weighted norms in both the state and the control spaces are introduced. Thus, for fixed symmetric and positive definite matrices $Q_x \in \mathbb{R}^{n_x \times n_x}$ and $Q_i \in \mathbb{R}^{n_i \times n_i}$, $i = 1, \ldots, m$, we let:

$$\begin{array}{rcl} Q_u &=& \mathrm{diag}\{Q_1, \cdots, Q_m\} \in \mathbb{R}^{n_u \times n_u} \\ Q_{ii} &=& \mathrm{diag}\{Q_i, Q_i\} \in \mathbb{R}^{2n_i \times 2n_i} \\ Q_{uu} &=& \mathrm{diag}\{Q_u, Q_u\} \in \mathbb{R}^{2n_u \times 2n_u} \\ Q_z &=& \mathrm{diag}\{Q_x, Q_u, Q_u\} \in \mathbb{R}^{(n_x+2n_u) \times (n_x+2n_u)} \\ Q_w &=& \mathrm{diag}\{Q_x, \cdots, Q_x\} \in \mathbb{R}^{mn_x \times mn_x} \\ \mathscr{Q}_\chi &=& \mathrm{diag}\{Q_x, Q_u\} \in \mathbb{R}^{(n_x+n_u) \times (n_x+n_u)}. \end{array}$$

The controller is sequentially designed starting from the lower level. Nevertheless, for the sake of clarity, the upper level controller is initially presented. To this end, however, the essential properties the low level controllers should fulfill need to be specified first.

Thus, according to Assumption 2.5, let $\tau \geq \ell$ and suppose that, $\forall i = 1, \ldots, m$ and $\forall \hat{u}_i \in \mathscr{U}_i$, a controller

$$\delta\nu_i(h) = \varphi_i(\delta\zeta_i(h), \hat{u}_i)$$

has been designed for system (1.16) so that the equilibrium $\hat{\zeta}_i(\hat{u}_i)$ is reached at most in τ steps. Under this assumption, two properties hold: first, $\alpha_i(k-1)$ and $u_i(k-1)$ determine the state of the i-th low level system at time $h=\tau k$ (in fact, by equation (1.15), $\zeta_i(\tau k)=\Gamma_i\alpha_i(k-1)u_i(k-1)$). Hence, the information on the state of the low level system is condensed in $\mathscr{A}_1(k-1)u(k-1)\in\mathbb{R}^{n_u}$, which is available at the high level controller. Notice that such a vector has in general lower dimension than the one collecting the ζ_i's (i.e., $n_u\leq\sum_{i=1}^{m}n_{\zeta_i}$). Secondly, $w_i(k)$ is univocally determined by $\alpha_i(k-1)u_i(k-1)$ and $u_i(k)$. In fact: both the state of the i-th subsystem at time $h=\tau k$ and the control law for the successive τ steps, since also $u_i(k)$ is given, are known; thus the output sequence $\tilde{u}_i(h)$ for $h=\tau k,\tau k+1,\dots,\tau(k+1)-1$ is known, as well as $w_i(k)$ by equations (1.24) and (1.6). Therefore,

$$w_i(k)=f_i(\alpha_i(k-1)u_i(k-1),u_i(k)),$$

where $f_i:\mathscr{U}_i\times\mathscr{U}_i\to\mathbb{R}^{n_x}$ only depends on the model of the i-th actuator and on the family of the low level controllers $\{\varphi_i(\cdot,\hat{u}_i)\}_{\hat{u}_i\in\mathscr{U}_i}$. Then, $w(k)$ in system (1.12) is given by

$$w(k)=f(\mathscr{A}_1(k-1)u(k-1),u(k)), \tag{1.25}$$

$f:\mathscr{U}\times\mathscr{U}\to\mathbb{R}^{mn_x}$ being defined in an obvious manner in terms of the f_i's. Notice that the components $w_i(k)$ such that $\alpha_i(k)=0$ have no relevance.

Assumption 3 *The function f is available to the high level controller, as well as a pair $\gamma_{\mathrm{d}}(i)\geq 0$ and $\delta_i>0$, $\forall i=1,\dots,m$, such that $\forall y_1,y_2\in\mathscr{U}_i$, $\|y_1\|_{Q_i}\leq\delta_i$ and $\|y_2\|_{Q_i}\leq\delta_i$, the following gain condition holds:*

$$\|f_i(y_1,y_2)\|_{Q_x}\leq\gamma_{\mathrm{d}}(i)\left\|\begin{bmatrix} y_1\\ y_2\end{bmatrix}\right\|_{Q_{ii}}. \tag{1.26}$$

The design of the low level controllers and the evaluation of the corresponding pairs $\gamma_{\mathrm{d}}(i)\geq 0$ and $\delta_i>0$ such that condition (1.26) holds will be thoroughly discussed in Section 1.3.2.
An alternative approach, to be considered when the function f is not known to the upper level, is also discussed in Remark 7 of Section 1.3.1.

1.3.1 The high level controller

In view of Assumption 3, the design of the high level controller is carried out by considering w as a disturbance satisfying a gain condition of the type $\|w\|_{Q_w}\leq\gamma_{\mathrm{d}}\|z\|_{Q_z}$ (for a suitable $\gamma_{\mathrm{d}}>0$). Thus, the small-gain approach is taken to derive a controller that guarantees robust stability with respect to such disturbance w. Clearly, the only relevant components of w are those such that $\alpha_i\neq 0$. Hence, for a given $\gamma_{\mathrm{d}}>0$, it is useful to consider the boolean function $\Phi_{\gamma_{\mathrm{d}}}:\mathscr{X}\times\left(\{0,1\}^m\times\mathscr{U}\right)\to\{0,1\}$ defined as follows: letting $\chi=[\begin{array}{cc} x & \mu\end{array}]'$,

$$\Phi_{\gamma_{\mathrm{d}}}\left(\chi,(\alpha,u)\right)=\begin{cases} 1 \quad \text{if} \quad \|\mathscr{A}_2 f(\mu,u)\|_{Q_w}\leq\gamma_{\mathrm{d}}\left\|\begin{bmatrix} x\\ \mu\\ \mathscr{A}_1 u\end{bmatrix}\right\|_{Q_z} \\ 0 \quad \text{otherwise.}\end{cases} \tag{1.27}$$

In particular, according to equations (1.13) and (1.25),

$$\Phi_{\gamma_{\mathrm{d}}}\left(\chi(k),(\alpha(k),u(k))\right)=1 \quad\Longleftrightarrow\quad \|\mathscr{A}_2(k)w(k)\|_{Q_w}\leq\gamma_{\mathrm{d}}\|z(k)\|_{Q_z}. \tag{1.28}$$

An auxiliary switching-and-control law guaranteeing robust stability is constructed first. Then, an MPC state feedback controller is derived so that the corresponding closed loop system has both better performance and a larger region of attraction with respect to the auxiliary law.

The auxiliary law

We construct an auxiliary switching-and-control law taking the form

$$\begin{cases} \alpha(k) = \alpha_{\text{aux}} \quad \forall k \geq 0 \\ u(k) = K_{\text{aux}}\chi(k), \end{cases} \tag{1.29}$$

for some suitable $\alpha_{\text{aux}} \in \{0,1\}^m$ and $K_{\text{aux}} \in \mathbb{R}^{n_u \times (n_x+n_u)}$ (thus, the set of the active actuators associated with the auxiliary law is not switching).
To this end, it is useful to introduce the following notation: for a given $\alpha_{\text{aux}} \in \{0,1\}^m$, denote by $\mathscr{A}_{1\,\text{aux}}$ and $\mathscr{A}_{2\,\text{aux}}$ the matrices (1.8) associated to α_{aux}. Furthermore, we let $\check{n}_u = \sum_{i\in\mathscr{I}(\alpha_{\text{aux}})} n_i$ and $\check{m} = \#\mathscr{I}(\alpha_{\text{aux}})$ be the number of active actuators associated with the auxiliary law. Denote by $\check{\mathscr{A}} \in \mathbb{R}^{(n_x+\check{n}_u)\times(n_x+\check{n}_u)}$, $\check{\mathscr{B}}_1 \in \mathbb{R}^{(n_x+\check{n}_u)\times\check{n}_u}$ and $\check{\mathscr{B}}_2 \in \mathbb{R}^{(n_x+\check{n}_u)\times(\check{m}n_x)}$ the matrices obtained by $\mathscr{A}$, $\mathscr{B}_1$ and $\mathscr{B}_2$ removing the rows and the columns associated with the null components of α_{aux}. Define $\check{\chi} \in \check{\mathscr{X}} = \mathbb{R}^{n_x} \times \check{\mathscr{U}} \subset \mathbb{R}^{n_x+\check{n}_u}$, $\check{\mathscr{Q}}_\chi \in \mathbb{R}^{(n_x+\check{n}_u)\times(n_x+\check{n}_u)}$, $\check{Q}_u \in \mathbb{R}^{\check{n}_u\times\check{n}_u}$ and $\check{Q}_w \in \mathbb{R}^{\check{m}n_x\times\check{m}n_x}$ in the analogous way.

Definition 1 *A vector $\alpha_{\text{aux}} \in \{0,1\}^m$ is said to identify a* consistent *configuration of the actuators iff the corresponding pair $(\check{\mathscr{A}}, \check{\mathscr{B}}_1)$ is stabilizable.*

For any consistent configuration $\alpha_{\text{aux}} \in \{0,1\}^m$, consider $\gamma > 0$ such that there exists a symmetric and positive definite matrix $\check{\mathscr{P}} = \check{\mathscr{P}}(\alpha_{\text{aux}},\gamma) \in \mathbb{R}^{(n_x+\check{n}_u)\times(n_x+\check{n}_u)}$ satisfying the Riccati inequality

$$\begin{cases} -\check{\mathscr{P}} + \check{\mathscr{A}}'\check{\mathscr{P}}\check{\mathscr{A}} + \check{\mathscr{Q}}_\chi - \check{\mathscr{A}}'\check{\mathscr{P}}\begin{bmatrix} \check{\mathscr{B}}_1 & \check{\mathscr{B}}_2 \end{bmatrix}\mathscr{R}^{-1}\begin{bmatrix} \check{\mathscr{B}}_1 & \check{\mathscr{B}}_2 \end{bmatrix}'\check{\mathscr{P}}\check{\mathscr{A}} < 0 \\ \mathscr{R} = \begin{bmatrix} \check{\mathscr{B}}_1'\check{\mathscr{P}}\check{\mathscr{B}}_1 + \check{Q}_u & \check{\mathscr{B}}_1'\check{\mathscr{P}}\check{\mathscr{B}}_2 \\ \check{\mathscr{B}}_2'\check{\mathscr{P}}\check{\mathscr{B}}_1 & \check{\mathscr{B}}_2'\check{\mathscr{P}}\check{\mathscr{B}}_2 - \gamma^2\check{Q}_w \end{bmatrix} \\ \check{\mathscr{B}}_2'\check{\mathscr{P}}\check{\mathscr{B}}_2 - \gamma^2\check{Q}_w < 0 \end{cases} \tag{1.30}$$

and let

$$\check{\mathscr{K}}_{\text{aux}} = -\begin{bmatrix} I_{\check{n}_u} & 0_{\check{n}_u,\check{m}n_x} \end{bmatrix}\mathscr{R}^{-1}\begin{bmatrix} \check{\mathscr{B}}_1 & \check{\mathscr{B}}_2 \end{bmatrix}'\check{\mathscr{P}}\check{\mathscr{A}}.$$

Define $P \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_u)}$ and $K_{\text{aux}} \in \mathbb{R}^{n_u\times(n_x+n_u)}$ by adding null rows and columns to $\check{\mathscr{P}}$ and $\check{\mathscr{K}}_{\text{aux}}$ in correspondence with the null components of α_{aux}. Let

$$V_f(\chi) = \chi' P \chi. \tag{1.31}$$

For any $\rho > 0$, consider $\tilde{\Omega}_\rho = \{\chi \in \mathbb{R}^{n_x+n_u} \,|\, V_f(\chi) \leq \rho^2\} \subset \mathbb{R}^{n_x+n_u}$. By the construction of P, this definition imposes a constraint on the components of $\check{\chi}$ only. Hence, consider the set of indices $\{j_1,\ldots,j_{m-\check{m}}\} = \{1,\ldots,m\} \setminus \mathscr{I}(\alpha_{\text{aux}})$ corresponding to the inactive actuators and let

$$\Omega_\rho = \{\chi \in \tilde{\Omega}_\rho \,|\, u_{j_1} \in \mathscr{U}_{j_1},\ldots,u_{j_{m-\check{m}}} \in \mathscr{U}_{j_{m-\check{m}}}\} \subset \mathbb{R}^{n_x+n_u}. \tag{1.32}$$

The local robust stabilization properties of the resulting auxiliary switching-and-control law (1.29) are clarified by the following result.

Proposition 1 *Let $\alpha_{\text{aux}} \in \{0,1\}^m$ be a consistent configuration and define*

$$\gamma_{\text{d}}(\alpha_{\text{aux}}) = \max_{i\in\mathscr{I}(\alpha_{\text{aux}})} \gamma_{\text{d}}(i). \tag{1.33}$$

Let $\gamma > 0$ be such that $\gamma \cdot \gamma_{\rm d}(\alpha_{\rm aux}) < 1$ and assume that a positive definite solution $\check{\mathscr{P}} = \check{\mathscr{P}}(\alpha_{\rm aux}, \gamma) \in \mathbb{R}^{(n_x+\check{n}_u)\times(n_x+\check{n}_u)}$ exists for the Riccati inequality (1.30)*. Consider system* (1.12), (1.13) *under the corresponding switching-and-control law* (1.29) *and, accordingly, let $w(k)$ be given by equation* (1.25)*. Then, $\exists \rho > 0$ such that all the following properties hold for $\Omega_{\rm aux} = \Omega_\rho$:*

$$
\begin{cases}
\Omega_{\rm aux} \subseteq \mathscr{X}\,; & (1.34a)\\
\forall \chi \in \Omega_{\rm aux},\ K_{\rm aux}\chi \in \mathscr{U}\,; & (1.34b)\\
\forall \chi \in \Omega_{\rm aux},\ \Phi_{\gamma_{\rm d}(\alpha_{\rm aux})}\big(\chi, (\alpha_{\rm aux}, K_{\rm aux}\chi)\big) = 1\,; & (1.34c)\\
\forall \chi(k) \in \Omega_{\rm aux} \text{ it holds that} & \\
V_f(\chi(k+1)) - V_f(\chi(k)) \le -(\|z(k)\|^2_{Q_z} - \gamma^2 \|\mathscr{A}_{2\,\rm aux} w(k)\|^2_{Q_w})\,. & (1.34d)
\end{cases}
$$

In particular, $\Omega_{\rm aux}$ is positively invariant.

Proof. By the definition of Ω_ρ, we only need to focus on $\check{\chi}$. Thus, property (1.34a) is guaranteed by the choice of a sufficiently small $\rho > 0$ since $\check{\mathscr{P}} > 0$ and $\check{\mathscr{X}}$ is a neighborhood of the origin.
As far as property (1.34b) is concerned, similarly to the previous case, we only need to focus on $\check{\mathscr{K}}_{\rm aux}$. The property is then guaranteed by the choice of a sufficiently small $\rho > 0$ since $\check{\mathscr{P}} > 0$, $\check{\mathscr{U}}$ is a neighborhood of the origin and the control law $\check{u} = \check{\mathscr{K}}_{\rm aux}\check{\chi}$ is continuous.
Let us consider property (1.34c): we have to guarantee that $\forall \chi = [\ x \quad \mu\ ]' \in \Omega_{\rm aux}$, letting $u = K_{\rm aux}\chi$, one has

$$
\|\mathscr{A}_{2\,\rm aux} f(\mu, u)\|_{Q_w} \le \gamma_{\rm d}(\alpha_{\rm aux}) \left\| \begin{bmatrix} x \\ \mu \\ \mathscr{A}_{1\,\rm aux} u \end{bmatrix} \right\|_{Q_z}
$$

(notice that, by the definition of $K_{\rm aux}$, the pre-multiplication of u by $\mathscr{A}_{1\,\rm aux}$ could be omitted). We claim that this property is ensured if, $\forall i \in \mathscr{I}(\alpha_{\rm aux})$, one has $\|\mu_i\|_{Q_i} \le \delta_i$ and $\|u_i\|_{Q_i} \le \delta_i$. Then, similarly to the proof of properties (1.34a) and (1.34b), this is guaranteed by the choice of a sufficiently small $\rho > 0$. Let us prove the claim:

$$
\begin{aligned}
\|\mathscr{A}_{2\,\rm aux} f(\mu, u)\|^2_{Q_w} = {} & \textstyle\sum_{i \in \mathscr{I}(\alpha_{\rm aux})} \|f_i(\mu_i, u_i)\|^2_{Q_x} \le \\
\overset{(a)}{\le} {} & \textstyle\sum_{i \in \mathscr{I}(\alpha_{\rm aux})} \gamma_{\rm d}^2(i) \left\| \begin{bmatrix} \mu_i \\ u_i \end{bmatrix} \right\|^2_{Q_{ii}} \le \\
\overset{(b)}{\le} {} & \gamma_{\rm d}^2(\alpha_{\rm aux}) \textstyle\sum_{i \in \mathscr{I}(\alpha_{\rm aux})} \left\| \begin{bmatrix} \mu_i \\ u_i \end{bmatrix} \right\|^2_{Q_{ii}} \le \\
\le {} & \gamma_{\rm d}^2(\alpha_{\rm aux}) \left\| \begin{bmatrix} x \\ \mu \\ \mathscr{A}_{1\,\rm aux} u \end{bmatrix} \right\|^2_{Q_z},
\end{aligned}
$$

where inequalities (a) and (b) follow by (1.26) and (1.33), respectively.
The proof of property (1.34d) can be found in [27]: the case considered in this report is exactly the one in [27] because the auxiliary configuration is not switching.
Finally, the positive invariance of $\Omega_{\rm aux}$ is guaranteed by inequalities (1.34) and the small-gain condition $\gamma \cdot \gamma_{\rm d}(\alpha_{\rm aux}) < 1$. In fact: inequality (1.34b) ensures the satisfaction of the input constraint; inequalities (1.34c-d) and $\gamma \cdot \gamma_{\rm d}(\alpha_{\rm aux}) < 1$ ensure that $V_f(\chi(k+1)) \le V_f(\chi(k))$ (see also (1.28)), while

the components of $\chi(k+1)$ corresponding to the null rows and columns of P are zero by definition of K_{aux}. ■

Remark 3 *According to the result reported in [27], property* (1.34d) *holds under the more general assumption* $\|\mathscr{A}_{2\,\mathrm{aux}}w(k)\|_{Q_w} \leq \gamma_{\mathrm{d}}(\alpha_{\mathrm{aux}})\|z(k)\|_{Q_z}\ \forall k \in \mathbb{N}$ *(rather than for* $w(k)$ *given by equation* (1.25) *only).*

Remark 4 (The choice of the auxiliary configuration) *For a given consistent configuration* α_{aux}*, there exists a value* $\gamma_{\mathrm{inf}}(\alpha_{\mathrm{aux}})$ *such that, for* $\gamma < \gamma_{\mathrm{inf}}(\alpha_{\mathrm{aux}})$*, a solution to the Riccati inequality* (1.30) *does not exist. Namely,* $\gamma_{\mathrm{inf}}(\alpha_{\mathrm{aux}})$ *is the minimal* H_∞ *norm of the closed loop system which can be achieved with the configuration* α_{aux}*. Therefore, if*

$$\gamma_{\mathrm{inf}}(\alpha_{\mathrm{aux}}) \cdot \gamma_{\mathrm{d}}(\alpha_{\mathrm{aux}}) \geq 1,$$

then α_{aux} *is not a feasible choice for the auxiliary configuration. In order to find a feasible configuration, one has to cope with the following* trade-off*: a consistent configuration* $\tilde{\alpha}_{\mathrm{aux}}$ *such that* $\gamma_{\mathrm{d}}(\tilde{\alpha}_{\mathrm{aux}}) < \gamma_{\mathrm{d}}(\alpha_{\mathrm{aux}})$ *can be obtained by discarding the slowest actuators of the previously selected configuration* α_{aux} *(i.e., those characterized by the largest values of* $\gamma_{\mathrm{d}}(i)$*). Nevertheless, in so doing, one has* $\mathscr{I}(\tilde{\alpha}_{\mathrm{aux}}) \subset \mathscr{I}(\alpha_{\mathrm{aux}})$*, namely less degrees of freedom are available in the control design. This may result in the increase of the minimal closed loop* H_∞ *norm achievable with the configuration* $\tilde{\alpha}_{\mathrm{aux}}$*, i.e.,* $\gamma_{\mathrm{inf}}(\tilde{\alpha}_{\mathrm{aux}}) \geq \gamma_{\mathrm{inf}}(\alpha_{\mathrm{aux}})$*.*

The high level MPC controller

The region of attraction Ω_{aux} and the performance provided by the auxiliary control law (1.29) are now improved with the MPC approach.
The gain condition (1.26) holds only locally. Nevertheless, according to Assumption 3, the high level controller can predict the disturbance $\mathscr{A}_2(k)w(k)$, once $\alpha(k)$ and $u(k)$ have been fixed (see equation (1.25)). It is hence possible to evaluate the boolean function $\Phi_{\gamma_{\mathrm{d}}}$ (see equation (1.27)) and, according to (1.28), to assess whether the control actions predicted by the high level MPC can be tracked by the actuators as accurately as needed to guarantee that $\|\mathscr{A}_2(k)w(k)\|_{Q_w} \leq \gamma_{\mathrm{d}}\|z(k)\|_{Q_z}$. Consequently, the MPC algorithm can also exploit the whole range of validity of such a gain condition, rather than limiting its action to the local operativity set ensured by condition (1.26).
Thus, in view of Assumption 3, it is possible to set up an MPC algorithm, where *sequences* of predicted switching-and-control values are considered, instead of *policies* as it would be more usual in a robust setting. It turns out that the desired robustness level, and hence stabilization, is guaranteed by only considering the H_∞ performance index in the cost function to be minimized. This is an unusual approach which stems from the peculiarity of the considered control problem. In fact, although the disturbance $w(k)$ can be conveniently predicted, it depends on both the current state $\chi(k)$ (through its component accounting for $\mathscr{A}_1(k-1)u(k-1)$) and on the control action $u(k)$ to be undertaken.
In details, let $N_p \in \mathbb{N}$, $N_p \geq 1$, be the length of the prediction horizon and $N_c \in \mathbb{N}$, $N_c \leq N_p$, be the length of the control horizon. Define by

$$\begin{cases} \mathscr{S}(k,N_c) &= \begin{bmatrix} \alpha(k) & \alpha(k+1) & \cdots & \alpha(k+N_c-1) \end{bmatrix} \\ \mathscr{F}(k,N_c) &= \begin{bmatrix} u(k) & u(k+1) & \cdots & u(k+N_c-1) \end{bmatrix}, \end{cases} \tag{1.35}$$

where $\alpha(k+j) \in \{0,1\}^m$ and $u(k+j) \in \mathscr{U}$, the vectors of the predicted switching-and-control signals to be processed by the MPC algorithm. At any time instant k, the control problem consists of solving the following optimization problem:

$$\min_{\substack{\mathscr{S}(k,N_c)\\ \mathscr{F}(k,N_c)}} J(\chi(k),\mathscr{S},\mathscr{F},N_p), \tag{1.36}$$

$$J(\chi(k),\mathscr{S},\mathscr{F},N_p) = \sum_{j=0}^{N_p-1}(\|z(k+j)\|^2_{Q_z} - \gamma^2\|\mathscr{A}_2(k+j)w(k+j)\|^2_{Q_w}) + V_f(\chi(k+N_p)),$$

subject to:

i) system (1.12), (1.13) under the switching-and-control signals (1.35) and, for $j = N_c,\ldots,N_p-1$,

$$\alpha(k+j) = \alpha_{\text{aux}}$$
$$u(k+j) = K_{\text{aux}}\chi(k+j),$$

and where $w(k+j)$ is given by equation (1.25);

ii) the state and control constraints

$$\forall j = N_c,\ldots,N_p-1, \quad u(k+j)\in\mathscr{U}$$
$$\forall j = 0,\ldots,N_p-1, \quad \begin{cases} \chi(k+j)\in\mathscr{X} \\ \Phi_{\gamma_d(\alpha_{\text{aux}})}\big(\chi(k+j),(\alpha(k+j),u(k+j))\big) = 1; \end{cases}$$

iii) the terminal constraint $\chi(k+N_p)\in\Omega_{\text{aux}}$.

If $\left(\bar{\mathscr{S}}(\chi(k),N_c),\bar{\mathscr{F}}(\chi(k),N_c)\right)$ is the optimal solution of this problem, according to the Receding Horizon principle, define the MPC switching-and-control law as

$$\begin{cases} \alpha(\chi(k)) = \bar{\alpha}(k) \\ u(\chi(k)) = \bar{u}(k). \end{cases} \tag{1.37}$$

Theorem 1 *Under the assumptions of Proposition 1, let* $\mathscr{X}^{MPC}(N_c,N_p)$ *be the set of states* $\chi\in\mathscr{X}$ *such that problem* (1.36) *admits a solution. Then* $\forall N_p\geq 1$ *and* $\forall N_c\leq N_p$*, one has:*

$\imath$) $\Omega_{\text{aux}}\subseteq\mathscr{X}^{MPC}(N_c,N_p)$.

Moreover, the following properties hold for the closed-loop system (1.12), (1.37), (1.25)*:*

$\imath\imath$) $\mathscr{X}^{MPC}(N_c,N_p)$ *is a positively invariant set;*

$\imath\imath\imath$) *The origin is a locally exponentially stable equilibrium point with region of attraction* $\mathscr{X}^{MPC}(N_c,N_p)$.

Proof. The theorem is proved for $N_c\geq 1$ (the case $N_c=0$ easily follows by Proposition 1).
$\imath$) $\Omega_{\text{aux}}\subseteq\mathscr{X}^{MPC}(N_c,N_p)$ because, by properties (1.34), the auxiliary law is feasible for $\chi\in\Omega_{\text{aux}}$ and Ω_{aux} is positively invariant.
$\imath\imath$) If $\chi(k)\in\mathscr{X}^{MPC}(N_c,N_p)$, then there exist $\bar{\mathscr{S}}$ and $\bar{\mathscr{F}}$ such that $\chi(k+N_p)\in\Omega_{\text{aux}}$. Thus, at time $k+1$, consider the following switching-and-control signal:

$$\begin{cases} \hat{\mathscr{S}}(k+1,N_c) &= \begin{bmatrix} \bar{\alpha}(k+1) & \cdots & \bar{\alpha}(k+N_c-1) & \alpha_{\text{aux}} \end{bmatrix} \\ \hat{\mathscr{F}}(k+1,N_c) &= \begin{bmatrix} \bar{u}(k+1) & \cdots & \bar{u}(k+N_c-1) & K_{\text{aux}}\chi(k+N_c) \end{bmatrix}. \end{cases}$$

This policy is still feasible for $\chi(k+1)$, and hence $\mathscr{X}^{MPC}(N_c,N_p)$ is a positively invariant set, because under the auxiliary switching-and-control law Ω_{aux} is positively invariant.
iii) Let

$$V(\chi(k),N_c,N_p) = J(\chi(k),\bar{\mathscr{S}},\bar{\mathscr{F}},N_c,N_p)$$

be the optimal performance starting from $\chi(k)$. We shall prove the validity of the following properties:

$$\begin{cases} \forall \chi \in \mathscr{X}^{MPC}(N_c,N_p), \text{ it holds that} \\ V(\chi,N_c,N_p) \geq (1-\gamma^2\cdot\gamma_{\text{d}}^2(\alpha_{\text{aux}}))\|\chi\|^2_{\mathscr{Q}_\chi}; & (1.38\text{a}) \\ \forall \chi \in \Omega_{\text{aux}}, \text{ it holds that } V(\chi,N_c,N_p) \leq \dfrac{\lambda_{\max}(P)}{\lambda_{\min}(\mathscr{Q}_\chi)}\|\chi\|^2_{\mathscr{Q}_\chi}; & (1.38\text{b}) \\ \forall \chi(k) \in \mathscr{X}^{MPC}(N_c,N_p), \text{ it holds that} \\ V(\chi(k+1),N_c,N_p) - V(\chi(k),N_c,N_p) \leq -(1-\gamma^2\cdot\gamma_{\text{d}}^2(\alpha_{\text{aux}}))\|z(k)\|^2_{Q_z}. & (1.38\text{c}) \end{cases}$$

Since $\|z(k)\|^2_{Q_z} \geq \|\chi(k)\|^2_{\mathscr{Q}_\chi}$, then, in view of the well known Theorem III.2 proved in [22], the stability result follows.

As far as inequality (1.38a) is concerned, for $\chi(k) \in \mathscr{X}^{MPC}(N_c,N_p)$, one has

$$\begin{aligned} V(\chi(k),N_c,N_p) = {} & J(\chi(k),\bar{\mathscr{S}},\bar{\mathscr{F}},N_c,N_p) = \\ = {} & \textstyle\sum_{j=0}^{N_p-1}(\|z(k+j)\|^2_{Q_z} - \gamma^2\|\mathscr{A}_2(k+j)w(k+j)\|^2_{Q_w}) + V_f(\chi(k+N_p)) \geq \\ \overset{(a)}{\geq} {} & (1-\gamma^2\cdot\gamma_{\text{d}}^2(\alpha_{\text{aux}}))\|z(k)\|^2_{Q_z} \geq \\ \geq {} & (1-\gamma^2\cdot\gamma_{\text{d}}^2(\alpha_{\text{aux}}))\|\chi(k)\|^2_{\mathscr{Q}_\chi}, \end{aligned}$$

where inequality (a) holds because, by constraints **ii)** and relation (1.28), it holds that, $\forall j = 0,\ldots,N_p-1$, $\|z(k+j)\|^2_{Q_z} - \gamma^2\|\mathscr{A}_2(k+j)w(k+j)\|^2_{Q_w} \geq (1-\gamma^2\cdot\gamma_{\text{d}}^2(\alpha_{\text{aux}}))\|z(k+j)\|^2_{Q_z} \geq 0$.
In order to prove inequality (1.38b), we first show that, if $\chi \in \mathscr{X}^{MPC}(N_c,N_p)$, then

$$V(\chi,N_c+1,N_p+1) \leq V(\chi,N_c,N_p). \tag{1.39}$$

Indeed, consider the switching-and-control signal

$$\begin{cases} \widetilde{\mathscr{S}}(k,N_c+1) &= \begin{bmatrix} \bar{\mathscr{S}}(\chi(k),N_c) & \alpha_{\text{aux}} \end{bmatrix} \\ \widetilde{\mathscr{F}}(k,N_c+1) &= \begin{bmatrix} \bar{\mathscr{F}}(\chi(k),N_c) & K_{\text{aux}}\chi(k+N_c) \end{bmatrix}, \end{cases}$$

then

$$\begin{gathered} J(\chi(k),\widetilde{\mathscr{S}}(k,N_c+1),\widetilde{\mathscr{F}}(k,N_c+1),N_c+1,N_p+1) = \\ = \textstyle\sum_{j=0}^{N_p}(\|z(k+j)\|^2_{Q_z} - \gamma^2\|\mathscr{A}_2(k+j)w(k+j)\|^2_{Q_w}) + V_f(\chi(k+N_p+1)) = \\ = \textstyle\sum_{j=0}^{N_p-1}(\|z(k+j)\|^2_{Q_z} - \gamma^2\|\mathscr{A}_2(k+j)w(k+j)\|^2_{Q_w}) + V_f(\chi(k+N_p+1)) + \\ + V_f(\chi(k+N_p)) - V_f(\chi(k+N_p)) + (\|z(k+N_p)\|^2_{Q_z} - \gamma^2\|\mathscr{A}_2(k+N_p)w(k+N_p)\|^2_{Q_w}). \end{gathered}$$

Since $\chi(k+N_p) \in \Omega_{\text{aux}}$ and the value of the output $z(k+N_p)$ is obtained with the auxiliary control law used at time $k+N_p$, using inequality (1.34d), one has

$$\begin{aligned} &J(\chi(k), \widetilde{\mathscr{S}}(k,N_c+1), \widetilde{\mathscr{F}}(k,N_c+1), N_c+1, N_p+1) \leq \\ &\leq \sum_{j=0}^{N_p-1}(\|z(k+j)\|^2_{Q_z} - \gamma^2\|\mathscr{A}_2(k+j)w(k+j)\|^2_{Q_w}) + V_f(\chi(k+N_p)) = \\ &= V(\chi(k),N_c,N_p). \end{aligned}$$

Consequently, $V(\chi(k),N_c+1,N_p+1) \leq V(\chi(k),N_c,N_p)$, thus proving the (1.39).
Now, $\forall \chi(k) \in \Omega_{\text{aux}}$,

$$\begin{aligned} &V(\chi(k),N_c,N_p) \overset{(b)}{\leq} V(\chi(k),0,N_p-N_c) = \\ &= \sum_{j=0}^{N_p-N_c-1}(\|z(k+j)\|^2_{Q_z} - \gamma^2\|\mathscr{A}_2(k+j)w(k+j)\|^2_{Q_w}) + V_f(\chi(k+N_p-N_c)) \leq \\ &\overset{(c)}{\leq} \sum_{j=0}^{N_p-N_c-1}(V_f(\chi(k+j)) - V_f(\chi(k+j+1))) + V_f(\chi(k+N_p-N_c)) = \\ &= V_f(\chi(k)) \leq \lambda_{\max}(P)\|\chi(k)\|^2 \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(\mathscr{Q}_\chi)}\|\chi\|^2_{\mathscr{Q}_\chi}, \end{aligned}$$

where inequality (b) follows by iterating the (1.39) (notice that $\forall N_c \leq N_p$, $\Omega_{\text{aux}} \subseteq \mathscr{X}^{MPC}(N_c,N_p)$) and inequality (c) holds in view of inequality (1.34d) (which can be applied because the length of the control horizon is 0 and, over the prediction horizon, the system evolves under the auxiliary law).
Finally, let us prove inequality (1.38c): observe that

$$V(\chi(k),N_c,N_p) = \|z(k)\|^2_{Q_z} - \gamma^2\|\mathscr{A}_2(k)w(k)\|^2_{Q_w} + V(\chi(k+1),N_c-1,N_p-1),$$

then, in view of (1.39),

$$V(\chi(k+1),N_c,N_p) - V(\chi(k),N_c,N_p) \leq -(\|z(k)\|^2_{Q_z} - \gamma^2\|\mathscr{A}_2(k)w(k)\|^2_{Q_w}).$$

Hence, by constraint **ii)**,

$$V(\chi(k+1),N_c,N_p) - V(\chi(k),N_c,N_p) \leq -(1-\gamma^2\cdot\gamma_{\text{d}}^2(\alpha_{\text{aux}}))\|z(k)\|^2_{Q_z}.$$

■

Remark 5 *The considered control problem has two main features: the* hierarchical *structure of the controller and the optimal* control structure selection *issue. This reflects on the coexistence of continuous and discrete variables, u and α, and on the Mixed Integer Quadratic Programming nature of the optimization problem* (1.36)*. If one is only interested in the hierarchical face of the problem, then the following approach can be taken: first, a feasible auxiliary configuration α_{aux} is fixed (see Remark 4 in Section 1.3.1) and $\alpha(k) \equiv \alpha_{\text{aux}}$ is considered; then problem* (1.36) *is solved by optimizing only over the control sequences $\mathscr{F}(k,N_c)$. Such a problem reduces to a constrained Quadratic Programming, whose computational burden is significantly cut down, and Theorem 1 still guarantees the exponential stabilization.*

Remark 6 (On the decoupling among the levels of the hierarchy) *The function f in equation* (1.25)*, if explicitly available, represents the only knowledge on the low level subsystems needed by the high level controller. We shall see in Section 1.3.2 that, when the low level systems are unconstrained and controlled with a linear law, f is simply represented by a matrix.*

However, the low level controller proposed in Section 1.3.2 relies, in part, on an MPC algorithm. Consequently, for practical purposes, such controllers $\{\varphi_i(\cdot,\hat{u}_i)\}_{i=1,\ldots,m;\,\hat{u}_i\in\mathscr{U}_i}$*, and hence the function* f*, are only implicitly known. This means that, to evaluate* $w(k+j)$ *for* $j=0,\ldots,N_p-1$*, the high level MPC algorithm should include a simulator of the low level subsystems. Hence, in the proposed approach, the two levels of the hierarchy appear to be coupled. Actually, as Remark 9 of Section 1.3.3 will clarify, such a coupling issue only holds during the transient behavior and, even in this period, simulating the low level subsystems can be partially avoided.*

Remark 7 (Min-max approach for complete decoupled control synthesis) *An alternative approach to the design of the high level controller can be pursued in case that neither the low level controllers nor the function* f *in equation* (1.25) *are available to the high level controller, whilst a constant* $\overline{\gamma}_{\mathrm{d}}\geq 0$ *is known by the high level controller such that,* $\forall i=1,\ldots,m$ *and* $\forall y_1,y_2\in\mathscr{U}_i$*, one has*

$$\|f_i(y_1,y_2)\|_{Q_x}\leq\overline{\gamma}_{\mathrm{d}}\left\|\begin{bmatrix} y_1\\ y_2\end{bmatrix}\right\|_{Q_{ii}} \tag{1.40}$$

(namely, condition (1.26) *holds globally rather than only locally). In this case, it is possible to completely decouple the design of the controllers at the two levels of the hierarchical structure. To this end, one may consider the term* $w(k)$ *as an unknown disturbance to be rejected. By inequality* (1.40)*, such a disturbance satisfies the gain condition* $\|\mathscr{A}_2(k)w(k)\|_{Q_w}\leq\overline{\gamma}_{\mathrm{d}}\|z(k)\|_{Q_z}$*. Thus, taking a robust control approach, the high level control algorithm can be modified by considering a min-max MPC formulation [28] and an equivalent result to Theorem 1 can be proved.*
Specifically, once a consistent auxiliary configuration has been fixed, an auxiliary control law can be obtained as in Proposition 1 (just replacing $\gamma_{\mathrm{d}}(\alpha_{\mathrm{aux}})$ *with* $\overline{\gamma}_{\mathrm{d}}$*). As for the MPC algorithm, the sequence* $\mathscr{D}(k,N_p)=\begin{bmatrix} w(k) & w(k+1) & \cdots & w(k+N_p-1)\end{bmatrix}$ *of the disturbances acting over the prediction horizon is introduced; switching-and-control* policies, *rather than* signals, *are considered (i.e.,* $\mathscr{S}(\chi(k),N_c)$ *and* $\mathscr{F}(\chi(k),N_c)$ *are vectors of feedback functions* $\alpha^{(j)}:\mathscr{X}\to\{0,1\}^m$ *and* $\kappa^{(j)}:\mathscr{X}\to\mathbb{R}^{n_u}$*,* $j=0,\ldots,N_c-1$*). Then the optimization problem* (1.36) *takes the min-max form*

$$\min_{\substack{\mathscr{S}(\chi(k),N_c)\\ \mathscr{F}(\chi(k),N_c)}}\ \max_{\mathscr{D}(k,N_p)} J(\chi(k),\mathscr{S},\mathscr{F},\mathscr{D},N_c,N_p), \tag{1.41}$$

under similar state and control constraints but condition

$$\Phi_{\gamma_{\mathrm{d}}(\alpha_{\mathrm{aux}})}\big(\chi(k+j),(\alpha(k+j),u(k+j))\big)=1$$

is replaced by the disturbance constraint

$$\|\mathscr{A}_2(k+j)w(k+j)\|_{Q_w}\leq\overline{\gamma}_{\mathrm{d}}\|z(k+j)\|_{Q_z},\quad \forall j=0,\ldots,N_p-1.$$

In view of Remark 3 in Section 1.3.1, the proof of the stability result is similar to the one of Theorem 1, only minor adjustments being needed (see also [47, 51]).
The main advantage of this approach consists in the possibility of almost completely decoupling the projects of the controllers at the high and at the low level (the high level controller needs only to know $\overline{\gamma}_{\mathrm{d}}$*). From a computational point of view, although the effort to simulate the low level subsystems is avoided, one has to undertake the min-max optimization* (1.41) *which is much more intensive.*
In order to follow this approach, one has to evaluate the weighted ℓ_2*-gain of each* f_i*, i.e.,*

$$\sup_{(y_1,y_2)\in\mathscr{U}_i^2\setminus\{(0,0)\}}\frac{\|f_i(y_1,y_2)\|_{Q_x}}{\left\|\begin{bmatrix} y_1\\ y_2\end{bmatrix}\right\|_{Q_{ii}}}.$$

This may be a difficult task, especially when the maps f_i's are associated with MPC controllers at the lower level: in this case, in fact, such maps are typically nonlinear and known only implicitly. It is instead a viable approach when the actuators are unconstrained and controlled with a linear law (see Section 1.3.2 below).

1.3.2 The low level controller

In the sequel we will use the following lemma:

Lemma 1 *Let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^n$, then $\forall x, y \in \mathbb{R}^n$ it holds that*

$$\|x-y\| \leq \sqrt{2}\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\|.$$

Proof. Indeed, $\forall x, y \in \mathbb{R}^n$, one has

$$\begin{aligned} \|x-y\|^2 - 2\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\|^2 &= \sum_{j=1}^n (x_j - y_j)^2 - 2\sum_{j=1}^n (x_j^2 + y_j^2) = \\ &= -\sum_{j=1}^n (x_j + y_j)^2 \leq 0. \end{aligned}$$

■

Let us go back to the design of the low level controllers. Recall that the aim is to design them so that, $\forall \hat{u}_i \in \mathscr{U}_i$, the equilibrium $\hat{\zeta}_i(\hat{u}_i)$ is reached at most in τ steps. We are also interested in the determination of a pair $\gamma_{\mathrm{d}}(i) \geq 0$ and $\delta_i > 0$ such that condition (1.26) is satisfied.
A controller having two parallel operation modes is proposed: at time $h = \tau k$, according to the value of the past and of the current references, one of the two modes is selected and is kept operating until the next long sampling time $h = \tau(k+1)$.

Mode 1: *Local linear controller*. If the norm of both the last and the current references $\hat{u}_i$'s is sufficiently small, the low level controller is a linear feedback law.

Mode 2: *MPC with approaching zero terminal state constraint*. Alternative to mode 1, an MPC algorithm is used which guarantees that the desired equilibrium is reached at most in τ steps.

Let us describe these controllers in details and let us specify their exact operational region, as well as the value of the corresponding gain $\gamma_{\mathrm{d}}(i)$ appearing in equation (1.26).

• **Mode 1:** First, a controller is defined under the assumption that no input and state constraints are considered (thus, $\zeta_i \in \mathbb{R}^{n_{\varsigma_i}}$ and $v_i \in \mathbb{R}^{n_i}$). This hypothesis will then be removed by suitably restricting the operational region of this controller.
Then, let $K_i \in \mathbb{R}^{n_i \times n_{\varsigma_i}}$ be such that $(F_i + G_i K_i)^\tau = 0$ (such a K_i exists in view of Assumptions 2.1 and 2.5). For any $\hat{u}_i \in \mathbb{R}^{n_i}$, consider the state feedback controller

$$\delta v_i = K_i \delta \zeta_i. \tag{1.42}$$

Let us analyze the gain condition (1.26) under this control law. To this end, we first determine the function f_i expressing the dependence of w_i from y_1 and y_2 (i.e., the dependence of $w_i(k)$ from the past reference $\hat{u}_i(\tau(k-1)) = \alpha_i(k-1)\overline{u}_i(\tau(k-1))$ and the new value $\overline{u}_i(\tau k)$ provided by the high level controller, respectively). It holds that w_i is a linear function of $y_1 - y_2$, more precisely: by equation (1.6), one has

$$\begin{aligned} w_i(k) = & \sum_{j=0}^{\tau-1} (A^f)^{\tau-j-1} b_i^f (u_i^f(\tau k + j) - y_2) \\ = & \sum_{j=0}^{\tau-1} (A^f)^{\tau-j-1} b_i^f \delta u_i(\tau k + j), \end{aligned}$$

where $\delta u_i(\tau k + j)$ is the output at time $\tau k + j$ of system (1.16) (with $\hat{u}_i = y_2$) under the control law (1.42). Since $(F_i + G_i K_i)^\tau = 0$, then $\zeta_i(\tau k) = \hat{\zeta}_i(y_1)$. At time τk, the new reference is y_2, thus

$$\delta\zeta_i(\tau k) = \hat{\zeta}_i(y_1) - \hat{\zeta}_i(y_2) = \Gamma_i(y_1 - y_2),$$

where matrix Γ_i is given in equation (1.15). It then follows that, letting

$$\mathscr{O}_i^{(\tau)} := \begin{bmatrix} H_i \\ H_i(F_i + G_i K_i) \\ \vdots \\ H_i(F_i + G_i K_i)^{\tau-1} \end{bmatrix} \quad \text{and} \quad \mathscr{R}_i^{(\tau)} := \begin{bmatrix} (A^f)^{\tau-1} b_i^f & \cdots & A^f b_i^f & b_i^f \end{bmatrix},$$

one has

$$w_i(k) = \mathscr{L}_i(y_1 - y_2), \tag{1.43}$$

where $\mathscr{L}_i = \mathscr{R}_i^{(\tau)} \mathscr{O}_i^{(\tau)} \Gamma_i \in \mathbb{R}^{n \times n_i}$. Hence,

$$\begin{aligned} \|w_i(k)\|_{Q_x} \leq & \sqrt{\lambda_{\max}(Q_x)}\, \|w_i(k)\| \leq \sqrt{\lambda_{\max}(Q_x)}\, \|\mathscr{L}_i\| \cdot \|y_1 - y_2\| \leq \\ \overset{(a)}{\leq} & \sqrt{2\lambda_{\max}(Q_x)}\, \|\mathscr{L}_i\| \left\| \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\| \leq \\ \leq & \sqrt{2\frac{\lambda_{\max}(Q_x)}{\lambda_{\min}(Q_i)}}\, \|\mathscr{L}_i\| \left\| \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\|_{Q_{ii}}, \end{aligned} \tag{1.44}$$

where $\|\mathscr{L}_i\| = \sqrt{\lambda_{\max}(\mathscr{L}_i' \mathscr{L}_i)}$ and inequality (a) holds by Lemma 1. Namely, in the absence of the input and state constraints (1.14), inequality (1.26) holds with

$$\gamma_{\text{d}}(i) = \sqrt{2\frac{\lambda_{\max}(Q_x)}{\lambda_{\min}(Q_i)}}\, \|\mathscr{L}_i\|.$$

Let us go back to the case where the state and input constraints (1.14) are taken into account. We determine $\delta_i > 0$ so that, if both $\|\hat{u}_i(\tau(k-1))\|_{Q_i} \leq \delta_i$ and $\|\hat{u}_i(\tau k)\|_{Q_i} \leq \delta_i$, then the linear controller (1.42) satisfies the constraints (1.17) and, switching among such references, the gain condition (1.44) is still valid. This study fixes the operational range of mode 1.

To this end, associated with the controller $\delta v_i = K_i \delta\zeta_i$, consider a quadratic Lyapunov function $V_i(\delta\zeta_i) = \delta\zeta_i' P_i \delta\zeta_i$: $\forall r \geq 0$, the set

$$\mathscr{N}_{\hat{\zeta}_i, r} = \{\zeta_i \in \mathbb{R}^{n_{\zeta_i}} \,|\, \|\zeta_i - \hat{\zeta}_i\|_{P_i} \leq r\}$$

is positively invariant for the unconstrained system.

For any equilibrium pair $(\hat{\zeta}_i, \hat{v}_i)$ belonging to the interior of $\mathscr{Z}_i \times \mathscr{V}_i$, there exists a sufficiently small $r(\hat{\zeta}_i, \hat{v}_i) > 0$ such that

$$\begin{cases} \mathscr{N}_{\hat{\zeta}_i, r(\hat{\zeta}_i, \hat{v}_i)} \subseteq \mathscr{Z}_i \\ \forall \zeta_i \in \mathscr{N}_{\hat{\zeta}_i, r(\hat{\zeta}_i, \hat{v}_i)},\ K_i(\zeta_i - \hat{\zeta}_i) \in \mathscr{V}_i - \hat{v}_i. \end{cases} \tag{1.45}$$

Namely, under constraints (1.17), the controller (1.42) is still well-defined within $\mathcal{N}_{\hat{\zeta}_i, r(\hat{\zeta}_i, \hat{v}_i)}$ and the positive invariance of this set is guaranteed.
Take a sufficiently small neighborhood $\mathscr{Z}_0^{(i)}$ of $\zeta_i = 0$ so that

$$R = \inf_{\hat{\zeta}_i \in \mathscr{Z}_0^{(i)}} r(\hat{\zeta}_i, \hat{v}_i) > 0. \tag{1.46}$$

Consider the neighborhood $\mathscr{Z}_{0\star}^{(i)}$ of 0 defined by $\mathscr{Z}_{0\star}^{(i)} := \{\zeta_i \in \mathscr{Z}_0^{(i)} \mid \|\zeta_i\|_{P_i} \leq \frac{R}{2}\}$. The following property holds:

$$\forall \hat{\zeta}_i \in \mathscr{Z}_{0\star}^{(i)},\ \mathscr{Z}_{0\star}^{(i)} \subseteq \mathcal{N}_{\hat{\zeta}_i, R} \tag{1.47}$$

(in fact, for $\hat{\zeta}_i \in \mathscr{Z}_{0\star}^{(i)}$ and $\zeta_i \in \mathscr{Z}_{0\star}^{(i)}$, it holds that $\|\zeta_i - \hat{\zeta}_i\|_{P_i} \leq R$ – i.e., $\zeta_i \in \mathcal{N}_{\hat{\zeta}_i, R}$ – because $\|\zeta_i - \hat{\zeta}_i\|_{P_i} \leq \|\zeta_i\|_{P_i} + \|\hat{\zeta}_i\|_{P_i} \leq \frac{R}{2} + \frac{R}{2}$). Since $\hat{\zeta}_i$ is a continuous function of $\hat{u}_i$, then

$$\exists \delta_i > 0 \text{ such that, } \forall \hat{u}_i \in \mathscr{U}_i \text{ with } \|\hat{u}_i\|_{Q_i} \leq \delta_i, \text{ one has } \hat{\zeta}_i(\hat{u}_i) \in \mathscr{Z}_{0\star}^{(i)}. \tag{1.48}$$

Thus, if $y_1 = \hat{u}_i(\tau(k-1))$ and $y_2 = \hat{u}_i(\tau k)$ are such that $\|y_1\|_{Q_i} \leq \delta_i$ and $\|y_2\|_{Q_i} \leq \delta_i$, then $\hat{\zeta}_i(y_1) \in \mathscr{Z}_{0\star}^{(i)}$ and $\hat{\zeta}_i(y_2) \in \mathscr{Z}_{0\star}^{(i)}$. Property (1.47) then guarantees that

$$\hat{\zeta}_i(y_1) \in \mathcal{N}_{\hat{\zeta}_i(y_2), R}.$$

Namely, in view of (1.46) and (1.45), the former equilibrium point $\hat{\zeta}_i(y_1)$ lies inside the neighborhood of the new equilibrium point $\hat{\zeta}_i(y_2)$ where the linear controller (1.42) satisfies the constraints (1.17). Therefore, switching from a reference y_1 such that $\|y_1\|_{Q_i} \leq \delta_i$ to a reference y_2 such that $\|y_2\|_{Q_i} \leq \delta_i$, relation (1.44) still holds true.

To recap, ***mode 1*** of the low level controller ***is defined as follows:***

> *If* $\|\hat{u}_i(\tau(k-1))\|_{Q_i} \leq \delta_i$ *and* $\|\hat{u}_i(\tau k)\|_{Q_i} \leq \delta_i$*, where* δ_i *is given in* (1.48)*, then mode 1 is active for* $\tau k \leq h < \tau(k+1)$ *and the control law is the linear feedback* (1.42)*. Relation* (1.26) *holds with* $\gamma_{\mathrm{d}}(i) = \sqrt{2\frac{\lambda_{\max}(Q_x)}{\lambda_{\min}(Q_i)}}\|\mathscr{L}_i\|$.

In all the other cases, the controller operates according to mode 2 as described below.

• **Mode 2:** The control action is determined through an MPC algorithm. In order to ensure that the state has reached the desired equilibrium at the end of the prediction horizon, an approaching zero terminal state constraint is included in the algorithm.
Let $\hat{u}_i$ be the reference for the i-th actuator (1.2) on the time interval $\tau k \leq h < \tau(k+1)$ and, accordingly, consider system (1.16). At time $\tau k + l$, $l = 0, \ldots, \tau - 1$, let

$$\mathscr{C}_i(l) = \begin{bmatrix} \delta v_i(\tau k + l) & \delta v_i(\tau k + l + 1) & \cdots & \delta v_i(\tau k + \tau - 1) \end{bmatrix} \tag{1.49}$$

be the vector of the predicted controls to be processed by the MPC algorithm. The control problem consists of solving the following optimization problem:

$$\min_{\mathscr{C}_i(l)} J_i(\delta\zeta_i(\tau k + l), \mathscr{C}_i(l)), \tag{1.50}$$

$$J_i(\delta\zeta_i(\tau k + l), \mathscr{C}_i(l)) = \Big\| \sum_{j=0}^{\tau-1} (A^f)^{\tau-j-1} b_i^f \delta u_i(\tau k + j) \Big\|,$$

subject to:

i) system (1.16) under the control law (1.49);

ii) the state and control constraints

$$\begin{cases} \delta\zeta_i(\tau k+j) \in \mathscr{X}_i - \hat{\zeta}_i & \forall j = l+1,\dots,\tau k+\tau-1 \\ \delta v_i(\tau k+j) \in \mathscr{V}_i - \hat{v}_i & \forall j = l,\dots,\tau k+\tau-1; \end{cases}$$

iii) the terminal constraint $\delta\zeta_i(\tau(k+1)) = 0$.

If $\overline{\mathscr{C}_i}(l)$ is the optimal solution of this problem, according to the Receding Horizon principle, define the MPC control law as

$$\delta v_i(\tau k+l) = \overline{\delta v}_i(\tau k+l). \tag{1.51}$$

Remark 8 *Few remarks are in order:*

- *First notice that the cost function $J_i(\delta\zeta_i(\tau k+l),\mathscr{C}_i(l))$ includes also the past output values, i.e., $\delta u_i(\tau k+j)$ for $j=0,\dots,l$. This choice ensures the minimization of $\|w_i(k)\|$ (see equation* (1.6)*);*
- *The optimization problem* (1.50) *is feasible $\forall l = 0,\dots,\tau-1$. In fact, the feasibility for $l=0$ is guaranteed by Assumption 2.5 (and $\tau \geq \ell$), then the feasibility for $l=1,\dots,\tau-1$ immediately follows because, according to the approaching zero terminal state constraint algorithm, the length of the prediction horizon is reduced by one at each step;*
- *Since the length of the prediction horizon varies, the resulting control law* (1.51) *is time varying;*
- *By construction, the control law* (1.51) *ensures that $\delta\zeta_i(\tau(k+1)) = 0$ (i.e., $\zeta_i(\tau(k+1)) = \hat{\zeta}_i(\hat{u}_i)$).*

1.3.3 The overall system: convergence analysis

The following result provides the analysis of the overall control system behavior.

Theorem 2 *Under the assumptions of Theorem 1, consider the closed loop system* (1.1), (1.19), (1.22) *and* (1.23)*, where the upper level controller is defined in* (1.37) *and the lower level controller is given by* (1.42) *and* (1.51)*. Assume that, at time $h=0$, the internal state of the actuators* (1.2) *is at an equilibrium, i.e., according to equation* (1.15)*, $\zeta_{i0} = \Gamma_i u_{i0}$ for some $u_{i0} \in \mathscr{U}_i$, $i=1,\dots,m$: let $\mu_0 = \begin{bmatrix} u_{10} & u_{20} & \cdots & u_{m0} \end{bmatrix}$. Assume also that the MPC controller at the upper level is initialized with $\chi(0) = \begin{bmatrix} x(0) & \mu_0 \end{bmatrix} \in X^{MPC}(N_c,N_p)$. Then it holds that*

$$\begin{cases} \lim_{h\to+\infty} x^f(h) = 0 \\ \lim_{h\to+\infty} \zeta_i(h) = 0, \quad \forall i=1,\dots,m. \end{cases}$$

Proof. Since $\chi(0) \in X^{MPC}(N_c,N_p)$ then, by Theorem 1, it holds that $\lim_{k\to+\infty}\chi(k) = 0$. This means that

$$\begin{cases} \lim_{k\to+\infty} x(k) = 0 \\ \lim_{k\to+\infty} \mathscr{A}_1(k)u(k) = 0, \end{cases}$$

and, according to equation (1.20),

$$\lim_{h\to+\infty} \hat{u}_i(h) = 0 \quad \forall i = 1,\dots,m.$$

In particular, $\exists \bar{k} > 0$ such that $\forall k \geq \bar{k}$ and $\forall i = 1,\dots,m$, $\|\hat{u}_i(\tau k)\|_{Q_i} \leq \delta_i$. This implies that, $\forall h \geq \tau(\bar{k}+1)$, the actuators are controlled according to the linear law (1.42) (i.e., according to mode 1 of the low level controller). Since such feedback controllers indeed stabilize the low level systems, then the convergence to zero of the reference signal $\hat{u}_i(h)$ translates into the convergence to zero of the internal state of the actuators, namely $\lim_{h\to+\infty} \zeta_i(h) = 0$, $\forall i = 1,\dots,m$. Moreover, $\lim_{h\to+\infty} \tilde{u}_i(h) = 0$.
Finally, we have to prove that $\lim_{h\to+\infty} x^f(h) = 0$. To this end, it is sufficient to show that, $\forall l = 1,\dots,\tau-1$, it holds that $\lim_{k\to+\infty} \|x^f(\tau k + l)\| = 0$. Indeed, combining equations (1.1), (1.3), (1.5) and (1.8), one has

$$x^f(\tau k + l) = (A^f)^l x^f(\tau k) + \textstyle\sum_{j=0}^{l-1} (A^f)^{l-j-1} B^f \mathscr{A}_1(k)\tilde{u}(\tau k + j),$$

where $\tilde{u}(h) = \begin{bmatrix} \tilde{u}_1(h)' & \tilde{u}_2(h)' & \cdots & \tilde{u}_m(h)' \end{bmatrix}'$. Since $\|\mathscr{A}_1(k)\| \leq 1$, then, $\forall l = 1,\dots,\tau-1$, it holds that

$$\|x^f(\tau k + l)\| \leq \|(A^f)^l\| \cdot \|x^f(\tau k)\| + \sum_{j=0}^{l-1} \|(A^f)^{l-j-1} B^f\| \cdot \|\tilde{u}(\tau k + j)\|.$$

As $x^f(\tau k) = x(k)$, $\lim_{k\to+\infty} x(k) = 0$ and, $\forall i = 1,\dots,m$, $\lim_{h\to+\infty} \tilde{u}_i(h) = 0$, the thesis follows. ■

Remark 9 (Further comments on the decoupling issue) *Let us go back to Remark 6 of Section 1.3.1. A suitable combination of the result of Theorem 2 with the analysis of mode 1 of the low level controller (see Section 1.3.2) allows one to further analyze the decoupling issue in the project of the controllers at the two levels of the hierarchy.*
First, for a control sequence $\mathscr{F}(k,N_c)$ predicted by the high level MPC controller, in correspondence to the components $u_i(k+j)$ resulting in the activation of mode 1, the value of $w_i(k+j)$ is explicitly provided by equation (1.43). *Thus, the high level MPC algorithm only needs to know the matrix $\mathscr{L}_i$ of equation* (1.43) *and the simulation of the corresponding low level subsystem is not needed.*
Furthermore, it is shown in the proof of Theorem 2 that both $\lim_{k\to+\infty} \chi(k) = 0$ and $\lim_{h\to+\infty} \hat{u}_i(h) = 0$, $\forall i = 1,\dots,m$. This means that, not only all the actuators are definitively controlled according to mode 1, but also that $\exists \bar{k}'$ such that, $\forall k \geq \bar{k}'$, the control sequences $\mathscr{F}(k,N_c)$ processed by the algorithm for the solution of the optimization problem (1.36) *are all made up of values corresponding to the activation of mode 1.*

1.4 Example

The two layer hierarchical control discussed so far is now applied to the linearized model of a well-mixed, non-isothermal, continuous stirred-tank reactor (CSTR), borrowed from [11, 31] and schematically depicted in Figure 1.1. Three parallel irreversible elementary exothermic reactions of the form $A \xrightarrow{k_1} B$, $A \xrightarrow{k_2} U$ and $A \xrightarrow{k_3} R$, respectively, take place within the CSTR, where A represents the reactant species, B the desired product, U and R undesired byproducts, k_i's the reaction constants.
According to [11, 31], the manipulated signals are the rate of heat input Q, the inlet stream temperature T_{A0} and the inlet reactant concentration C_{A0}, while the state variables are the temperature of the reactor T and the concentrations of the species A and B, C_A and C_B, respectively, which have been supposed to be measurable.

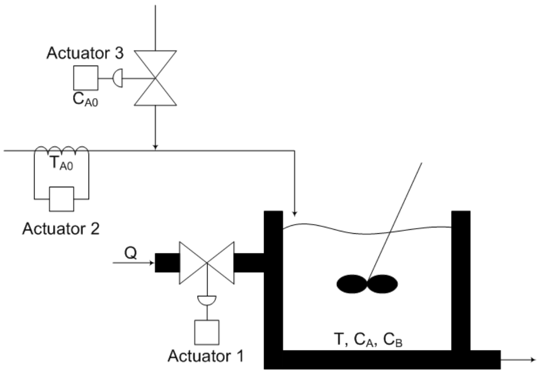

Figure 1.1: CSTR schematic plant.

The considered equilibrium condition is defined by the input quantities $\bar{Q} = 0$, $\bar{T}_{A0} = 300$ and $\bar{C}_{A0} = 4$ and the state values $\bar{T} = 300$, $\bar{C}_A = 4$ and $\bar{C}_B = 0$. Hence, letting

$$\begin{array}{rcl} u & = & \left[\begin{array}{ccc} Q - \bar{Q} & T_{A0} - \bar{T}_{A0} & C_{A0} - \bar{C}_{A0} \end{array} \right]' \\ x & = & \left[\begin{array}{ccc} T - \bar{T} & C_A - \bar{C}_A & C_B - \bar{C}_B \end{array} \right]' \end{array}$$

the corresponding continuous-time linearized model turns out to be

$$\dot{x} = \begin{bmatrix} -4.6337 & 1.3746 & 0 \\ -0.0017 & -5.0044 & 0 \\ 0.0017 & 0.0064 & -4.9980 \end{bmatrix} x + \begin{bmatrix} 0.0043 & 4.9980 & 0 \\ 0 & 0 & 4.9980 \\ 0 & 0 & 0 \end{bmatrix} u, \tag{1.52}$$

where the control variables are required to fulfill the following constraints:

$$|u_1| \leq 768, \quad |u_2| \leq 100, \quad |u_3| \leq 4. \tag{1.53}$$

Moreover, the actuators' dynamics described by the following transfer functions

$$G_{\text{act}\,1}(s) = \frac{20}{\frac{1}{700}s^2 + \frac{1}{9}s + 1}, \quad G_{\text{act}\,2}(s) = \frac{10}{\frac{1}{1200}s^2 + \frac{1}{15}s + 1}, \quad G_{\text{act}\,3}(s) = \frac{1}{\frac{1}{1500}s^2 + \frac{1}{18}s + 1} \tag{1.54}$$

have been considered and the constraints

$$|v_1| \leq 20, \quad |v_2| \leq 5, \quad |v_3| \leq 2 \tag{1.55}$$

on their input variables (i.e., the manipulated signals) have been defined.
The continuous-time models (1.52) and (1.54) have been sampled with $\Delta t = 0.01$ to obtain their discrete-time counterpart in the fast time scale; moreover, the corresponding versions in the slow time scale have been figured out by taking $\tau = 4$.
In the sequel, the design parameters for the numerical implementation of the proposed hierarchical controller are given. First of all, the following weighting matrices have been fixed:

$$Q_x = I_3, \qquad Q_u = \begin{bmatrix} 10^{-4} & 0 & 0 \\ 0 & 10^{-1} & 0 \\ 0 & 0 & 100 \end{bmatrix}. \tag{1.56}$$

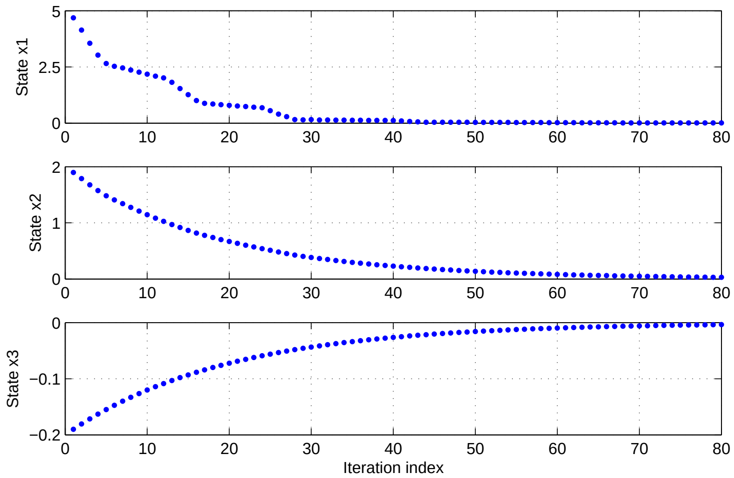

Figure 1.2: Sampled state variables $x_i^f(h)$ of model (1.52).

The choice of matrix Q_u stems from considering both the ease in acting on the rate of heat input (u_1), whose weighting entry has been fixed to $Q_1 = 10^{-4}$, and the difficulty in modifying the inlet reactant concentration (u_3), whose penalizing term has been frozen to $Q_3 = 100$. Conversely, the inlet stream temperature may be adjusted with a relatively manageable effort. Hence, the weighting term $Q_2 = 10^{-1}$ has been taken into account.
In addition, under the deadbeat controller (1.42), it turns out that $\max_{i=1,2,3} \gamma_{\mathrm{d}}(i) \simeq 0.0874$, so that

$$\alpha_{\mathrm{aux}} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

is a feasible configuration of active actuators for the auxiliary control law, with a viable attenuation level $\gamma = 11.44$. The Riccati inequality (1.30) has been solved accordingly, thus determining P and the auxiliary control gain K_{aux}. Afterwards, resorting to Proposition 1, the final set Ω_{aux} has been defined (corresponding to such a P, 100 is an admissible value for ρ). Finally, the MPC regulator at the upper level has been synthesized by picking the prediction and control horizons $N_p = N_c = 5$.
Figures 1.2-1.6 portray the simulation results, where the following initial conditions have been set up: $x^f(0) = \begin{bmatrix} 5 & 2 & -0.2 \end{bmatrix}'$ and $\zeta_i(0) = 0$, $i = 1,2,3$. Such results show a correctly stabilizing action of the plant to be controlled and the effectiveness of the algorithm implemented, which makes constraints (1.53) and (1.55) always fulfilled (note, in particular, Figure 1.6 where several saturation events on the low level manipulated variables arise). Moreover, each actuator's activity is consistent with the weighting matrices (1.56), as shown in Figures 1.3, 1.4 and 1.5, thereby the aforementioned different usage of the three actuators is highlighted. In addition, notice that the first actuator is always active, while the third one, even though often working, is required to achieve a small control action.

1.5 Conclusions

In this report, hierarchical control systems have been considered. A synthesis method based on MPC, and accounting also for the control structure selection issue, has been proposed. An unusual robust control approach has been carried out to cope with the discrepancies between the ideal control actions

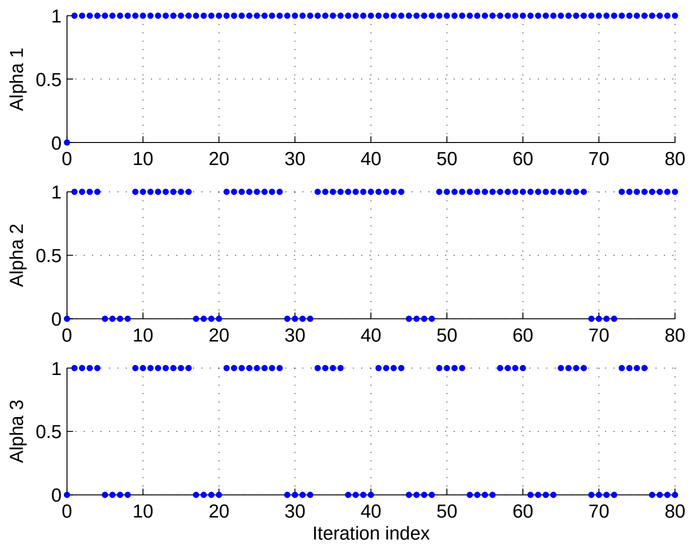

Figure 1.3: Boolean variables $\alpha_i^f(h)$ associated with each actuator.

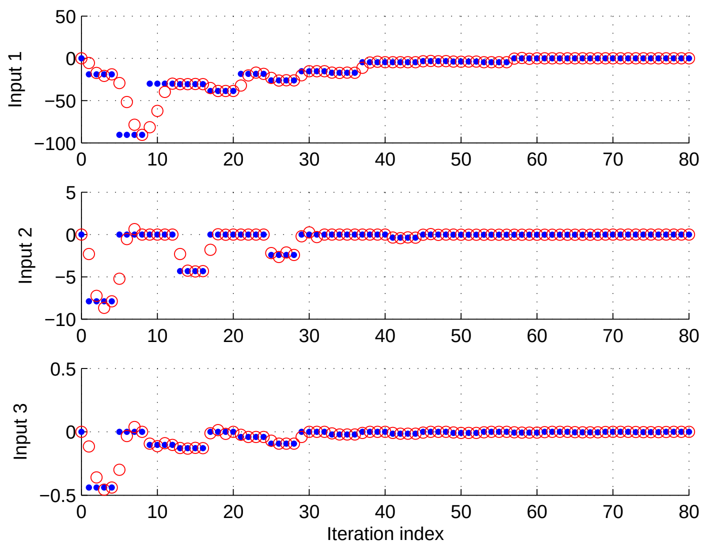

Figure 1.4: Comparison between the control references $\hat{u}_i(h)$ (dots) and the corresponding control actions $\tilde{u}_i(h)$ (circles) effectively provided by the actuators.

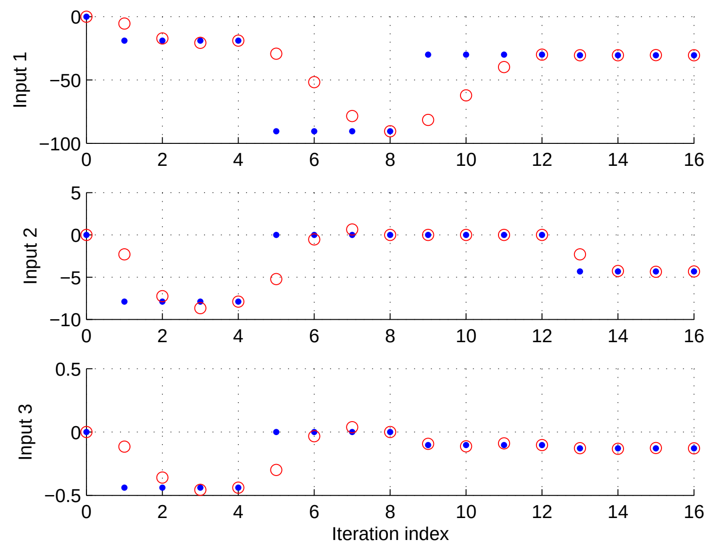

Figure 1.5: Zoom of the first 16 samples of Figure 1.4.

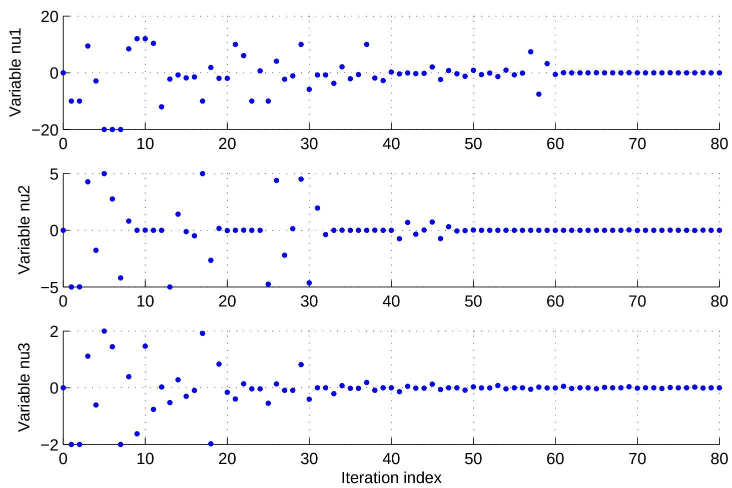

Figure 1.6: Actuators' manipulated variables $v_i(h)$.

computed by the high level controller and those effectively achieved by the low level actuators. The multi edge solution devised can simultaneously ensure the on-line selection of the optimal control configuration, deal with the mixed integer nature of the optimization problem, and attain convergence properties of the overall system.
Many extensions can be pursued, such as the investigation of hierarchical control structures made by more than two layers, or the application of this approach to "plug and play" design problems, see e.g., [32].

Chapter 2

Feasible cooperation distributed MPC scheme, based on game theory

The contents of this chapter have been developed by Jairo Espinosa, Felipe Valencia (Universidad Nacional de Colombia, Facultad de Minas), Bart De Schutter and Katerina Stankova (Delft University of Technology, Delft Center for Systems and Control).

2.1 Introduction

Large-scale systems are systems composed of several interacting components. Their operation is based on controllers that face different situations according to their own interests. For the control of large-scale systems, distributed and hierarchical control schemes based on model predictive controllers have been proposed, due to their ability to handle complex systems with hard input and state constraints [4, 45, 9, 38, 39, 40, 42], and for their ability to obtain a good performance starting from rather intuitive design principles and simple models [45].

Some approaches of hierarchical and distributed model predictive control are proposed in [4, 5, 6, 7, 10, 15, 16, 17, 21, 38, 50, 55, 56, 58, 62, 63, 68, 69, 71, 72, 73, 74]. However, the hierarchical approaches do not guarantee that the subsystems do not compete among themselves because each layer may take its own decisions without taking decisions of lower layers into account. The distributed approaches may force the subsystems to cooperate, often without taking into account whether the cooperative behavior gives some benefit to the subsystems, and might lead the subsystems to operating points in which the subsystems do not perceive any benefit.

In order to deal with these drawbacks of the distributed control schemes, it is possible to assume that the local controllers may "bargain" among themselves, and an agreement may be achieved. With such assumptions, the distributed model predictive control (MPC) problems can be reformulated as a n-persons cooperative games. The n-person cooperative game involves n individuals that can collaborate for mutual benefit. The individuals communicate among themselves in order to (jointly) decide which strategy is the best for each individual, based on the profit received for each of them by the cooperative behavior [36].

In this work, based on the Nash theories about the bargaining problem [36] and two-persons cooperative games [37] distributed model predictive control is analyzed as a game. In order to test the proposed control scheme, a chain of two reactors and one adiabatic flash separator is used as simulation testbed.

2.2 Distributed model predictive control

Consider the nonlinear system given by

$$\begin{aligned} \dot{x}(t) &= f_x(x(t),u(t)) \\ y(t) &= f_y(x(t),u(t)) \end{aligned} \tag{2.1}$$

where $f_x(.),\ f_y(.)$ are smooth C^1 functions, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^z$ denote the state, input, and output vector of the dynamical system (2.1).
Assume that at each time step k, the system (2.1) can be approximated by a discrete-time linear time-invariant system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{2.2}$$

In the following, we focus on the trajectories of the states x and their constraints. However, the method described in this work can easily be extended to the cases in which the output y is considered in the cost function and in the constraints.
The idea of MPC is to determine the control inputs for the system (2.1) at each time step k, by solving an optimal control problem over a prediction horizon N_p, based on the prediction of the behavior of the system (2.1) given by the model (2.2). Commonly, a quadratic cost function

$$\begin{aligned} L(\widetilde{x}(k),\widetilde{u}(k)) = &\sum_{t=0}^{N_\mathrm{p}-1} \left[x^T(k+t|k)Q_t x(k+t|k)\right] \\ &+ \sum_{t=0}^{N_\mathrm{u}} \left[u^T(k+t)R_t u(k+t)\right] \\ &+ x^T(k+N_\mathrm{p}|k)Px(k+N_\mathrm{p}|k) \end{aligned} \tag{2.3}$$

is used to measure the performance of the system. In (2.3), $x(k+t|k)$ denotes the predicted value of x at time step $k+t$ given the conditions at time step k, $u(k+t)$ denotes the control input u at time step $k+t$, $\widetilde{x}(k) = [x^T(k|k),\ldots,x^T(k+N_\mathrm{p}|k)]^T$, $\widetilde{u}(k) = [u^T(k),\ldots,u^T(k+N_\mathrm{u}),\ldots,u^T(k+N_\mathrm{p})]^T$, where $x(k|k) = x(k)$, and $u(k+t) = u(k+N_\mathrm{u}),$ for $t = N_\mathrm{u}+1,\ldots,N_\mathrm{p}-1$, $Q_t,\ R_t$ are diagonal matrices with positive diagonal elements, and $N_\mathrm{u} \leqslant N_\mathrm{p}$, $N_\mathrm{u},\ N_\mathrm{p}$ being the control and prediction horizon respectively, and P being the solution of the Lyapunov equation

$$A^T PA - P = -Q_{N_\mathrm{p}} \tag{2.4}$$

If (2.2) is stable (i.e., the norm of the eigenvalues of A is less than 1), and Q_{N_p} is positive definite, it follows that P is also positive definite [63].
By repetitively substituting the equation (2.2) for $x(k+t|k)$ into (2.3), and by using the control horizon constraint $u(k+t) = u(k+N_\mathrm{u}),$ for $t = N_\mathrm{u}+1,\ldots,N_\mathrm{p}-1$, the function $L(\widetilde{x}(k),\widetilde{u}(k))$ can be expressed as a quadratic function $\phi(\widetilde{u}(k);x(k))$, $x(k)$ being the value of the state vector at time step k. Then, the MPC problem can be formulated as follows

$$\begin{aligned} &\min_{\widetilde{u}(k)} \phi(\widetilde{u}(k);x(k)) \\ &\text{subject to: } \widetilde{u}(k) \in \Omega \end{aligned} \tag{2.5}$$

where Ω is the feasible set of control actions, determined by physical and operational limits of the system (2.1).

If the scale of the system (2.1) becomes large, the solution of optimization problem (2.5) becomes infeasible in real time. Hence, a distributed solution of (2.5) is necessary, and a decomposition of model (2.2) is also needed.
Assume that the state update equation (2.2) can be decomposed into M subsystems such that the behavior of each subsystem can be expressed as

$$x_i(k+1) = \sum_{j=1}^{M} [A_{ij}x_j(k) + B_{ij}u_j(k)] \tag{2.6}$$

where $x_i \in \mathbb{R}^{n_i}$, and $u_i \in \mathbb{R}^{m_i}$ denote the state and input vector of the subsystem i, $i = 1, \dots, M$, and A_{ij}, B_{ij} are submatrices of A, B. This model is also used in [4, 6, 8, 38, 63].
Based on the subsystem decomposition (2.6), the value of the state vector x_i at time step $k+1$ can be computed as

$$\begin{aligned} x_i(k+1) = {} & \begin{bmatrix} A_{i1} & \dots & A_{iM} \end{bmatrix} \begin{bmatrix} x_1(k) \\ \vdots \\ x_M(k) \end{bmatrix} \\ & + \begin{bmatrix} B_{i1} & \dots & B_{iM} \end{bmatrix} \begin{bmatrix} u_1(k) \\ \vdots \\ u_M(k) \end{bmatrix} \end{aligned} \tag{2.7}$$

and the function $L(\widetilde{x}(k), \widetilde{u}(k))$ can be expressed as

$$\begin{aligned} L(\widetilde{x}(k), \widetilde{u}(k)) = {} & \sum_{i=1}^{M} \left(\sum_{t=0}^{N_\mathrm{p}-1} x_i^T(k+t|k) Q_{it} x_i(k+t|k) \right. \\ & \left. + \sum_{t=0}^{N_\mathrm{u}} u_i^T(k+t) R_{it} u_i(k+t) + x_i^T(k+N_\mathrm{p}|k) P_i x(k+N_\mathrm{p}|k) \right) \end{aligned} \tag{2.8}$$

where $P_i = [P_{i,1}, \dots, P_{i,M}]$, and Q_{it}, R_{it} are submatrices of Q_t, R_t respectively. Note that in the distributed case, the terminal constraint relates the final value of the states of subsystem i, $x_i(k+N_\mathrm{p}|k)$, with the final value of the states of whole system $x(k+N_\mathrm{p}|k)$. This happens because (in general) P is a full matrix.
Repetitively substituting the equation (2.6) for $x_i(k+t|k)$ into (2.8) yields

$$L(\widetilde{x}(k), \widetilde{u}(k)) = \sum_{i=1}^{M} \phi_i(\widetilde{u}(k); x(k)) \tag{2.9}$$

$$\phi_i(\widetilde{u}(k); x(k)) = \widetilde{u}^T(k) Q_{uui} \widetilde{u}(k) + 2x^T(k) Q_{xui} \widetilde{u}(k) + x^T(k) Q_{xxi} x(k) \tag{2.10}$$

where $Q_{uui} \geq 0$, for $i = 1, \dots, M$. In the following, the term $x^T(k) Q_{xxi} x(k)$ will not be included into the cost function because the value of this term is independent of the value of the control inputs.
In (2.10) the arguments for $\phi_i(\widetilde{u}(k); x(k))$ indicate that the argument of ϕ_i is $\widetilde{u}(k)$ and $x(k)$ is parameter of ϕ_i. Clearly, ϕ_i is a positive-definite quadratic function of $\widetilde{u}(k)$ and thus it is convex on $\widetilde{u}(k)$.
Let Ω_i be the set of feasible control actions for $\widetilde{u}_i(k)$, where $\widetilde{u}_i(k) = [u_i^T(k), \dots, u_i^T(k+N_\mathrm{u})]^T$, determined by the physical and operational limits of subsystem i. Assume that $0 \in \Omega_i$ and that Ω_i convex

and independent of k for $i = 1,\dots,M$. Note that $\Omega = \Pi_{i=1}^{M}\Omega_i$. Then, the MPC problem (2.5) can be written as[1]

$$\begin{aligned} &\min_{\tilde{u}(k)} \sum_{i=1}^{M} \phi_i(\tilde{u}(k)) \\ &\text{subject to: } \tilde{u}_i(k) \in \Omega_i, \text{ for } i = 1,\dots,M \end{aligned} \tag{2.11}$$

The same problem is also considered in [24, 38, 65, 63].
Let $\sigma_r(\tilde{u}_i(k), \tilde{u}_{-i}(k)) = \phi_r(\tilde{u}(k))$, ($r = 1,\dots,M$), where

$$\tilde{u}_{-i}(k) = [\tilde{u}_1^T(k),\dots,\tilde{u}_{i-1}^T(k),\tilde{u}_{i+1}^T(k),\dots,\tilde{u}_M^T(k)]^T$$

$\tilde{u}_i(k) \in \Omega_i$, and $\tilde{u}_{-i}(k) \in \Omega_{-i}$, $\Omega_{-i} = \Omega_1 \times \dots \Omega_{i-1} \times \Omega_{i+1} \times \dots \times \Omega_M$. In order to solve the optimization problem (2.11) in a distributed fashion, two main approaches have been proposed in the literature, see [63, 64, 65]: the communication-based approach and the feasible-cooperation approach.
In the communication-based approach, each subsystem i solves its local optimization problem

$$\begin{aligned} &\min_{\tilde{u}_i(k)} \sigma_i(\tilde{u}_i(k), \tilde{u}_{-i}(k)) \\ &\text{subject to: } \tilde{u}_i(k) \in \Omega_i \end{aligned} \tag{2.12}$$

without carrying out a negotiation process with the other subsystems, assuming the control actions $\tilde{u}_{-i}(k)$ of the other subsystems are fixed.
In the feasible-cooperation approach, the local cost function $\phi_i(.)$ is replaced by a cost function that measures the systemwide impact of the local control inputs. This is used with the purpose of avoiding competition and to increase the cooperation among subsystems in a distributed model predictive control (DMPC) scheme. In [63, 65] the authors propose to use a convex combination of the controller objectives:

$$\phi_i(\tilde{u}(k)) = \sum_{r=1}^{M} w_r \sigma_r(\tilde{u}_i(k), \tilde{u}_{-i}(k)) \tag{2.13}$$

where $w_r > 0$, $\sum_{i=1}^{M} w_r = 1$ as a cost function of each subsystem, because it is the simplest choice for such objective function. Also, the subsystems must carry out a negotiation process in order to select the best control actions, with respect to the performance of the entire system. Let $\tilde{u}_{i,q}(k) = [u_{i,q}^T(k|k),\dots,u_{i,q}^T(k+N_\mathrm{u}|k)]^T$, where q denotes the iteration number at time step k of the negotiation process ($q = 1,\dots,q_\mathrm{max}$, q_max being the maximum number of iterations of the negotiation process). Let $\tilde{u}_{-i,q}(k) = [\tilde{u}_{1,q}^T(k),\dots,\tilde{u}_{i-1,q}^T(k),\tilde{u}_{i+1,q}^T(k),\dots,\tilde{u}_{M,q}^T(k)]^T$. Then, the optimization problem (2.11) can be solved in a cooperative way by computing the solution of the optimization problem [63, 65]

$$\begin{aligned} &\min_{\tilde{u}_{i,q}(k)} \sum_{r=1}^{M} w_r \sigma_r(\tilde{u}_{i,q}(k), \tilde{u}_{-i,q-1}(k)) \\ &\text{subject to: } \tilde{u}_{i,q}(k) \in \Omega_i \end{aligned} \tag{2.14}$$

In (2.14), the expression for $\sigma_r(\tilde{u}_{i,q}(k), \tilde{u}_{-i,q-1}(k))$ is given by

$$\begin{aligned} &\sigma_r(\tilde{u}_{i,q}(k), \tilde{u}_{-i,q-1}(k)) = \\ &\quad \begin{bmatrix} \tilde{u}_{i,q}(k) & \tilde{u}_{-i,q-1}(k) \end{bmatrix} \begin{bmatrix} H_{r1} & H_{r2} \\ H_{r3} & H_{r4} \end{bmatrix} \begin{bmatrix} \tilde{u}_{i,q}(k) \\ \tilde{u}_{-i,q-1}(k) \end{bmatrix} \\ &\quad + 2 \begin{bmatrix} f_{r1} & f_{r2} \end{bmatrix} \begin{bmatrix} \tilde{u}_{i,q}(k) \\ \tilde{u}_{-i,q-1}(k) \end{bmatrix} \end{aligned} \tag{2.15}$$

[1]For the sake of simplicity of notation we will not indicate the dependence of ϕ_i on $x(k)$ explicitly in the remainder of this chapter and thus write $\phi_i(u(k))$ instead $\phi_i(u(k);x(k))$.

where the matrices $H_{r1}, H_{r2}, H_{r3}, H_{r4}$ and the vectors f_{r1}, f_{r2} are obtained by permuting the elements of the matrix Q_{uur} and the element of the vector Q_{xur} respectively, in order to obtain the expression (2.15) for σ_r.
In the next section, theoretical concepts of game theory will be used in order to deal with the feasible-cooperation MPC (FC-MPC) problem as a decision problem in which the decisions of each subsystem affect the decisions of the other subsystems. The FC-MPC approach was selected because there exists evidence (e.g. the cases presented by [63]) that the communication-based MPC leads to unacceptable closed-loop performance or closed-loop instability.

2.3 FC-MPC as a game

A game is defined as the tuple $(N, \{\Omega_i\}_{i\in N}, \{\phi_i\}_{i\in N})$, where $N = \{1,\ldots,M\}$ is the set of players, Ω_i is a finite set of possible actions of player i, and $\phi_i : \Omega_1 \times \ldots \times \Omega_M \longrightarrow \mathbb{R}$ is the payoff function of the ith player [44].
Based on the definition of a game, the feasible cooperation model based predictive control problem can be defined as a tuple $G = (N, \{\Omega_i\}_{i\in N}, \{\phi_i\}_{i\in N})$, where $N = \{1,\ldots,M\}$ is the set of subsystems, Ω_i is the non-empty set of feasible control actions for subsystem i, and $\phi_i : \Omega_1 \times \ldots \times \Omega_M \longrightarrow \mathbb{R}$, where ϕ_i is the cost function of the ith subsystem. From this point of view, FC-MPC is a game in which the players are the subsystems, the actions are the control inputs, and the payoff of each subsystem is given by the value of its cost function. Moreover, in FC-MPC the subsystems can cooperate in order to obtain a common benefit. So, FC-MPC can be analyzed as a cooperative game.
Following the cooperative game theory introduced in [35, 37, 46], the formulation of the FC-MPC as a game is completed by introducing the concept of disagreement point. The disagreement point, $d_i(k)$, at time step k, is defined as the benefit that the ith player receives where no agreement is achieved among the players. In the case of FC-MPC, the disagreement point can be interpreted as the outcome of each subsystem if it decides not to cooperate with the other subsystems. Thus, the disagreement point of subsystem i is given by $d_i(k) = \phi_i(\widetilde{u}^{\mathrm{d}}(k))$, where $\widetilde{u}^{\mathrm{d}}(k)$ are the control inputs solving the following optimization problem

$$\begin{aligned} &\min_{\widetilde{u}_i(k)} \max_{\widetilde{u}_{-i}(k)} \phi_i(\widetilde{u}(k)) \\ \text{subject to: } &\widetilde{u}_i(k) \in \Omega_i \\ &\widetilde{u}_{-i}(k) \in \Omega_{-i} \end{aligned} \tag{2.16}$$

Note that the optimization problem (2.16) defines the worst case for subsystem i. Then, $d_i(k)$ is the best benefit that the ith subsystem can achieve given the worst case.
According to [36, 37], the solution of the cooperative game associated with the DMPC problem (2.11) can be computed as the solution of the optimization problem [14, 46]

$$\begin{aligned} &\max_{\widetilde{u}(k)} \Pi_{i=1}^{M} \left[d_i(k) - \phi_i(\widetilde{u}(k))\right]^{w_i} \\ \text{subject to: } &d_i(k) > \phi_i(\widetilde{u}(k)), \text{ for } i = 1,\ldots,M \\ &\widetilde{u}_i(k) \in \Omega_i, \text{ for } i = 1,\ldots,M \end{aligned} \tag{2.17}$$

where $d_i(k)$ is the disagreement point of subsystem i at time step k. The solution of the maximization problem (2.17) is called *asymmetric Nash bargaining solution*. The product $\Pi_{i=1}^{M}(d_i(k) - \phi_i(\widetilde{u}(k))^{w_i}$ is called *asymmetric Nash product*. Thus, the asymmetric Nash solution assigns to a bargaining game

the point of the feasible set dominated by the disagreement outcome ($d_i > \phi_i(\widetilde{u}(k))$), where the product of the profit functions of the players is maximized [46]. This kind of solution is used because it is possible to demonstrate that optimization problem (2.17) satisfies the four axioms proposed by Nash for the bargaining solutions of cooperative games [46].
The maximization problem (2.17) can rewritten equivalently as

$$\begin{aligned} &\max_{\widetilde{u}(k)} \sum_{i=1}^{M} w_i \log\left[d_i(k) - \phi_i(\widetilde{u}(k))\right] \\ \text{subject to: } & d_i(k) > \phi_i(\widetilde{u}(k)),\ \text{for } i = 1,\ldots,M \\ & \widetilde{u}_i(k) \in \Omega_i,\ \text{for } i = 1,\ldots,M \end{aligned} \tag{2.18}$$

Thus, (2.17) can be solved in a distributed fashion by solving (2.18) with the similar approach as (2.14). So, the local optimization problem for subsystem i is given by the maximization problem

$$\begin{aligned} &\max_{\widetilde{u}_i(k)} \sum_{r=1}^{M} w_r \log\left[d_r(k) - \sigma_r(\widetilde{u}_i(k), \widetilde{u}_{-i}(k))\right] \\ \text{subject to: } & d_r(k) > \sigma_r(\widetilde{u}_i(k), \widetilde{u}_{-i}(k)),\ \text{for } r = 1,\ldots,M \\ & \widetilde{u}_i(k) \in \Omega_i \end{aligned} \tag{2.19}$$

In the next section, we propose a negotiation model to solve the FC-MPC game.

2.4 Negotiation model

A negotiation model consists of a sequence of steps whose outcome is the solution of the game in a cooperative or non-cooperative fashion. The negotiation model proposed in this work is based on the algorithm proposed by [37] for two-persons cooperative games.
Before introduce the negotiation model we should introduce some notation that will be used later in this section. Recall that $\sigma_r(\widetilde{u}_i(k), \widetilde{u}_{-i}(k)) = \phi_r(\widetilde{u}(k))$, and in particular that $\sigma_i(\widetilde{u}_i(k), \widetilde{u}_{-i}(k)) = \phi_i(\widetilde{u}(k))$. Then, optimization problem (2.16) can be written as

$$\begin{aligned} &\min_{\widetilde{u}_i(k)} \max_{\widetilde{u}_{-i}(k)} \sigma_i(\widetilde{u}_i(k), \widetilde{u}_{-i}(k)) \\ \text{subject to: } & \widetilde{u}_i(k) \in \Omega_i \\ & \widetilde{u}_{-i}(k) \in \Omega_{-i} \end{aligned} \tag{2.20}$$

Let $\widetilde{u}^*_{-i}(k) = \arg\max_{\widetilde{u}_{-i}(k)} \sigma_i(\widetilde{u}_i(k), \widetilde{u}_{-i}(k))$. Let $\widetilde{u}^{\mathrm{d}}_i(k) = \arg\min_{\widetilde{u}_i(k)} \sigma_i(\widetilde{u}_i(k), \widetilde{u}^*_{-i}(k))$. Let $\widetilde{u}_0(k) = [\widetilde{u}^T_{1,0}(k), \ldots, \widetilde{u}^T_{M,0}(k)]^T$ denote the initial condition for solving (2.20) at time step k. Then, at each time step k, at each iteration $q, q = 1,\ldots,q_{\max}$ ($q_{\max}$ being the maximum number of iterations allowed to subsystem i, $i = 1,\ldots,M$ for solving (2.19) in a cooperative way), the proposed steps to solve the FC-MPC game are:

1. Let $d_i(k)$ denote the disagreement point of subsystem i at time step k. Then, given the initial conditions, $x(k)$, all subsystems compute their disagreement points $d_i(k)$ according to (2.20) in a separated way.

2. After computing the disagreement points, each subsystem sends its disagreement point to the other subsystems.

3. Each subsystem solves the optimization problem (2.19). If (2.19) is feasible, let $\widetilde{u}^*_{i,q}(k)$ be an optimal solution (so it satisfies the constraints, i.e., $d_r(k) > \sigma_r(\widetilde{u}^*_{i,q}(k), \widetilde{u}_{-i,q-1}(k))$, for $r = 1,\ldots,M$). If (2.19) is not feasible, subsystem i decides not to cooperate. In this step, if $q = 1$, then $\widetilde{u}^{\mathrm{d}}_i(k)$ is considered as initial condition for subsystem i, for solving (2.19). Otherwise, $\widetilde{u}_{i,q-1}(k)$ is considered as initial condition for subsystem i, for solving (2.19).

4. The subsystems that decide to cooperate update their control actions by a convex combination $\widetilde{u}_{i,q}(k) = w_i\widetilde{u}^*_{i,q}(k) + (1-w_i)\widetilde{u}_{i,q-1}(k)$. The subsystems that decide not to cooperate select their control actions equal to $\widetilde{u}_{i,q}(k) = w_i\widetilde{u}^{\mathrm{d}}_i(k) + (1-w_i)\widetilde{u}_{i,q-1}(k)$.

5. Each subsystem sends its control actions to the other subsystems. If $\|\widetilde{u}_{i,q}(k) - \widetilde{u}_{i,q-1}(k)\| \leqslant \xi$ ($\xi > 0$) for all subsystems, or $q = q_{\max}$, or the maximum allowable time for the computation of the optimal control input $\widetilde{u}^*(k) = [\widetilde{u}^{*T}_1(k),\ldots,\widetilde{u}^{*T}_M(k)]^T$ is reached, the first element of the control sequence $\widetilde{u}_{i,q}(k)$ is applied and each subsystem returns to step 1. Else, each subsystem returns to step 3.

At time step $k+1$ the initial conditions for subsystem i for solving (2.20) are determined by the shifted control sequence $\widetilde{u}_{i,0}(k+1) = [u^{*T}_{i,q_{\mathrm{end}}}(k+1,k),\ldots,u^{*T}_{i,q_{\mathrm{end}}}(k+N_{\mathrm{u}},k),0]^T$, where $u^{*T}_{i,q_{\mathrm{end}}}(k+1,k)$ denote the optimal value of the control inputs for subsystem i at iteration q_{end} at the time step $k+1$ given the conditions at time step k.
For the proposed negotiation model, properties like convexity, convergence and stability are topics for further research. However, in the next section we present simulation results using this negotiation model for the systemwide control of a plant consisting of a chain of two continuous stirred tank reactors followed by a non-adiabatic flash separator. From these simulation results it is possible to conclude that the trajectories of the states of (2.1) converges to the origin, and that (2.1) is stable under the proposed control scheme.

2.5 Application: Two-reactor chain with a flash separator

The example of a chain of two continuous stirred tank reactors (CSTRs) followed by a non-adiabatic flash separator was taken from [65]. All the simulations was performed using Matlab software. For solving the optimization problem (2.19) the active-set algorithm provided by the *fmincon* function of the optimization toolbox of Matlab was used. The description of the system is presented below.
Consider a plant with two CSTRs followed by a non-adiabatic flash separator. In each of the CSTRs, the desired product B is produced by through the irreversible first order reaction $A \xrightarrow{k_1} B$, k_1 being the Arrhenius constant of the reaction. An undesirable side reaction $B \xrightarrow{k_2} C$ results in the consumption of B and in the production of the unwanted side product C (here, k_2 is the Arrhenius constant of this reaction). The product stream from CSTR-2 is sent to a non-adiabatic flash separator to separate the excess of A from the product B and the side product C.
It is assumed that reactant A has the highest volatility and is the predominant component in the vapor phase in the flash separator. A fraction of the vapor phase is purged and the remaining stream rich in A is condensed and recycled back to CSTR-1. The model of the plant is given by

1. Reactor 1:

$$\begin{aligned}
\frac{dH_{\rm r}}{dt} &= \frac{1}{\rho_{\rm r}A_{\rm r}}\left[F_0 + D - F_{\rm r}\right] \\
\frac{dx_{\rm Ar}}{dt} &= \frac{1}{\rho_{\rm r}A_{\rm r}H_{\rm r}}\left[F_0(x_{\rm A0}-x_{\rm Ar}) + D(x_{\rm Ad}-x_{\rm Ar})\right] \\
&\quad - k_{\rm 1r}x_{\rm Ar} \\
\frac{dx_{\rm Br}}{dt} &= \frac{1}{\rho_{\rm r}A_{\rm r}H_{\rm r}}\left[F_0(x_{\rm B0}-x_{\rm Br}) + D(x_{\rm Bd}-x_{\rm Br})\right] \\
&\quad + k_{\rm 1r}x_{\rm Ar} - k_{\rm 2r}x_{\rm Br} \\
\frac{dT_{\rm r}}{dt} &= \frac{1}{\rho_{\rm r}A_{\rm r}H_{\rm r}}\left[F_0(T_0-T_{\rm r}) + D(T_{\rm d}-T_{\rm r})\right] \\
&\quad - \frac{1}{C_{\rm p}}\left[k_{\rm 1r}x_{\rm Ar}\Delta H_1 + k_{\rm 2r}x_{\rm Br}\Delta H_2\right] \\
&\quad + \frac{Q_{\rm r}}{\rho_{\rm r}A_{\rm r}C_{\rm p}H_{\rm r}}
\end{aligned} \tag{2.21}$$

2. Reactor 2:

$$\begin{aligned}
\frac{dH_{\rm m}}{dt} &= \frac{1}{\rho_{\rm m}A_{\rm m}}\left[F_{\rm r} + F_1 - F_{\rm m}\right] \\
\frac{dx_{\rm Am}}{dt} &= \frac{1}{\rho_{\rm m}A_{\rm m}H_{\rm m}}\left[F_{\rm r}(x_{\rm Ar}-x_{\rm Am}) + F_1(x_{\rm A1}-x_{\rm Am})\right] \\
&\quad - k_{\rm 1m}x_{\rm Am} \\
\frac{dx_{\rm Bm}}{dt} &= \frac{1}{\rho_{\rm m}A_{\rm m}H_{\rm m}}\left[F_{\rm r}(x_{\rm Br}-x_{\rm Bm}) + F_1(x_{\rm B1}-x_{\rm Bm})\right] \\
&\quad + k_{\rm 1m}x_{\rm Am} - k_{\rm 2m}x_{\rm Bm} \\
\frac{dT_{\rm m}}{dt} &= \frac{1}{\rho_{\rm m}A_{\rm m}H_{\rm m}}\left[F_{\rm r}(T_{\rm r}-T_{\rm m}) + F_1(T_0-T_{\rm m})\right] \\
&\quad - \frac{1}{C_{\rm p}}\left[k_{\rm 1m}x_{\rm Am}\Delta H_1 + k_{\rm 2m}x_{\rm Bm}\Delta H_2\right] \\
&\quad + \frac{Q_{\rm m}}{\rho_{\rm m}A_{m}C_{\rm p}H_{\rm m}}
\end{aligned} \tag{2.22}$$

3. Non-adiabatic flash:

$$\begin{aligned}
\frac{dH_{\rm b}}{dt} &= \frac{1}{\rho_{\rm b}A_{\rm b}}\left[F_{\rm m} - F_{\rm b} - D - F_{\rm p}\right] \\
\frac{dx_{\rm Ab}}{dt} &= \frac{1}{\rho_{\rm b}A_{\rm b}H_{\rm b}}\big[F_{\rm m}(x_{\rm Am}-x_{\rm Ab}) \\
&\quad - (D+F_{\rm p})(x_{\rm Ad}-x_{\rm Ab})\big] \\
\frac{dx_{\rm Bb}}{dt} &= \frac{1}{\rho_{\rm b}A_{\rm b}H_{\rm b}}\big[F_{\rm m}(x_{\rm Bm}-x_{\rm Bb}) \\
&\quad + (D+F_{\rm p})(x_{\rm Bd}-x_{\rm Bb})\big] \\
\frac{dT_{\rm b}}{dt} &= \frac{1}{\rho_{\rm b}A_{\rm b}H_{\rm b}}\left[F_{\rm m}(T_{\rm m}-T_{\rm b})\right] + \frac{Q_{\rm b}}{\rho_{\rm b}A_{\rm b}C_{\rm p}H_{\rm b}}
\end{aligned} \tag{2.23}$$

Table 2.1: Values of the parameters used in the simulations of the two CSTRs chain followed by a non-adiabatic flash system

Parameter	Units	Value
ρ_{r} ρ_{m} ρ_{b}	$[kg/m^3]$	0.15
α_{A}		3.5
α_{B}		1.1
α_{C}		0.5
k_1^*	$[s^{-1}]$	0.02
k_2^*	$[s^{-1}]$	0.018
A_{r}	$[m^2]$	0.3
A_{m}	$[m^2]$	3
A_{b}	$[m^2]$	5
T_0	$[K]$	313
T_{d}	$[K]$	313
C_{p}	$[KJ/(kgK)]$	25
x_{A0}		1
x_{B0} x_{C0}		0
x_{A1}		1
x_{B1} x_{C1}		0
ΔH_1	$[KJ/kg]$	-40
ΔH_2	$[KJ/kg]$	-50
E_1/R E_2/R	$[K]$	150
k_{r}	$[kg/s^{-1}m^{-1/2}]$	2.5
k_{m}	$[kg/s^{-1}m^{-1/2}]$	2.5
k_{b}	$[kg/s^{-1}m^{-1/2}]$	1.5

where, $F_{\mathrm{r}} = k_{\mathrm{r}}\sqrt{H_{\mathrm{r}}}$, $F_{\mathrm{m}} = k_{\mathrm{m}}\sqrt{H_{\mathrm{m}}}$, $F_{\mathrm{b}} = k_{\mathrm{b}}\sqrt{H_{\mathrm{b}}}$, $k_{1\mathrm{r}} = k_1^* \exp^{\frac{-E_1}{RT_{\mathrm{r}}}}$, $k_{2\mathrm{r}} = k_2^* \exp^{\frac{-E_2}{RT_{\mathrm{r}}}}$, $k_{1\mathrm{m}} = k_1^* \exp^{\frac{-E_1}{RT_{\mathrm{m}}}}$, $k_{2\mathrm{m}} = k_2^* \exp^{\frac{-E_2}{RT_{\mathrm{m}}}}$, $x_{\mathrm{Cr}} = 1 - x_{\mathrm{Ar}} - x_{\mathrm{Br}}$, $x_{\mathrm{Cm}} = 1 - x_{\mathrm{Am}} - x_{\mathrm{Bm}}$, $x_{\mathrm{Cb}} = 1 - x_{\mathrm{Ab}} - x_{\mathrm{Bb}}$, $\Sigma = \alpha_{\mathrm{A}} x_{\mathrm{Ab}} + \alpha_{\mathrm{B}} x_{\mathrm{Bb}} + \alpha_{\mathrm{C}} x_{\mathrm{Cb}}$, $x_{\mathrm{Ad}} = \frac{\alpha_{\mathrm{A}} x_{\mathrm{Ab}}}{\Sigma}$, $x_{\mathrm{Bd}} = \frac{\alpha_{\mathrm{B}} x_{\mathrm{Bb}}}{\Sigma}$, $x_{\mathrm{Cd}} = \frac{\alpha_{\mathrm{C}} x_{\mathrm{Cb}}}{\Sigma}$.

For the system (2.21) to (2.23), the manipulated variables are the feed flow rates F_0, F_1, the cooling duties $Q_{\mathrm{r}}, Q_{\mathrm{m}}, Q_{\mathrm{b}}$, and the recycle flow rate D. The measured variables are the level of liquid in the reactors $H_{\mathrm{r}}, H_{\mathrm{m}}, H_{\mathrm{b}}$, the exit mass fractions of A and B $x_{\mathrm{Ar}}, x_{\mathrm{Br}}, x_{\mathrm{Am}}, x_{\mathrm{Bm}}, x_{\mathrm{Ab}}, x_{\mathrm{Bb}}$, and the temperature of the reactors $T_{\mathrm{r}}, T_{\mathrm{m}}, T_{\mathrm{b}}$. The controlled variables are the liquid level of the reactors and the temperatures of the reactors.

With the purpose of applying the proposed distributed control scheme, each reactor and the flash separator were considered as subsystems. Then linearizing the model of each subsystem (equations (2.21) to (2.23)) around an operating point, and discretizing the resulting linear time-invariant models, a model defined by (2.6) for each subsystem is obtained. The values of the parameters used in the simulations and the operation point are in Tables 2.1 and 2.2 ([65]).

For the simulations, the constraints of the variables of the system were: $0.9F_{0\mathrm{o}} \leqslant F_0 \leqslant 1.1F_{0\mathrm{o}}$, $0.96F_{1\mathrm{o}} \leqslant F_1 \leqslant 1.04F_{1\mathrm{o}}$ for the feed flow rates of the CSTRs, and $0.1D_{\mathrm{o}} \leqslant D \leqslant 1.1D_{\mathrm{o}}$ for the recycle flow rate, and $-6 \leqslant Q_{\mathrm{r}} \leqslant 6$, $-6 \leqslant Q_{\mathrm{m}} \leqslant 6$, $-6 \leqslant Q_{\mathrm{b}} \leqslant 6$ for the cooling duty of the CSTRs and the adiabatic flash separator.

The performance of the proposed control scheme is evaluated when a set-point change corresponding

Table 2.2: Operation point used to linearize the two CSTRs chain followed by a non-adiabatic flash model

Variable	Units	Value
H_{ro}	$[m]$	180
H_{mo}	$[m]$	190
H_{bo}	$[m]$	5.2185
$F_{0\mathrm{o}}$	$[kg/s]$	2.667
$F_{1\mathrm{o}}$	$[kg/s]$	1.067
D_{o}	$[kg/s]$	30.74
Q_{ro}	$[KJ/s]$	0
Q_{mo}	$[KJ/s]$	0
Q_{bo}	$[KJ/s]$	0

to a 15.79 percent increase in the level of H_{m} is made at time $k = 200s$, and a set point change corresponding to a 5.56 percent decrease in the level of H_{r} is made at time $k = 400s$. Figs. 2.1 to 2.3 show the performance of the proposed control scheme for these set-point changes. In this simulation the prediction horizon $N_{\mathrm{p}} = 25$, the control horizon $N_{\mathrm{u}} = 10$, the weight of each subsystem $w = 0.33$, the sample time $T_{\mathrm{s}} = 10s$, and the maximum iterations per time step $q_{\max} = 5$.

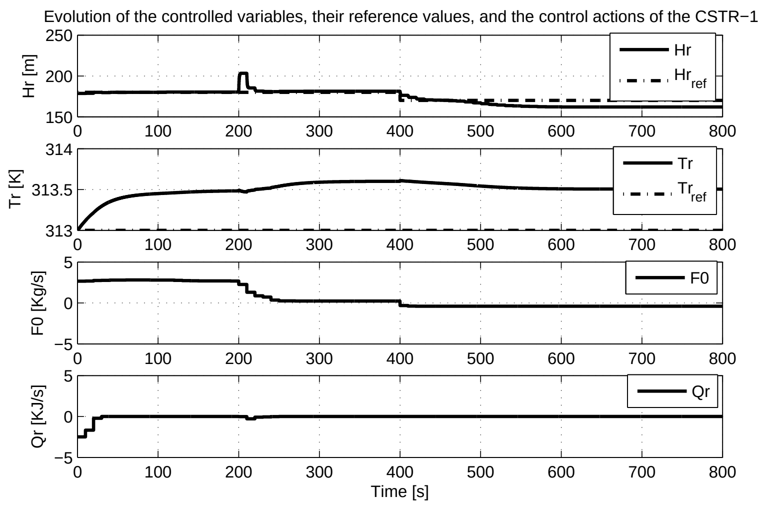

Figure 2.1: Evolution of the level and the temperature of the CSTR-1, their reference values, and the control inputs.

From Figs. 2.1 to 2.3 it is possible to conclude that the proposed control scheme stabilizes the closed-loop system. As a response to the set-point change of H_{m}, CSTR-1 and CSTR-2 jointly decide to decrease and increase their feed flow rates, respectively, in order to softly drive the system to the new desired operating point, while the flash separator decides to increase the recycle flow in order to regulate its level and the level of the CSTR-1. This indicates a cooperative behavior among the MPC controllers. In Fig. 2.4 the computational time incurred by the computation of the solution of the FC-MPC problem as a game is presented.

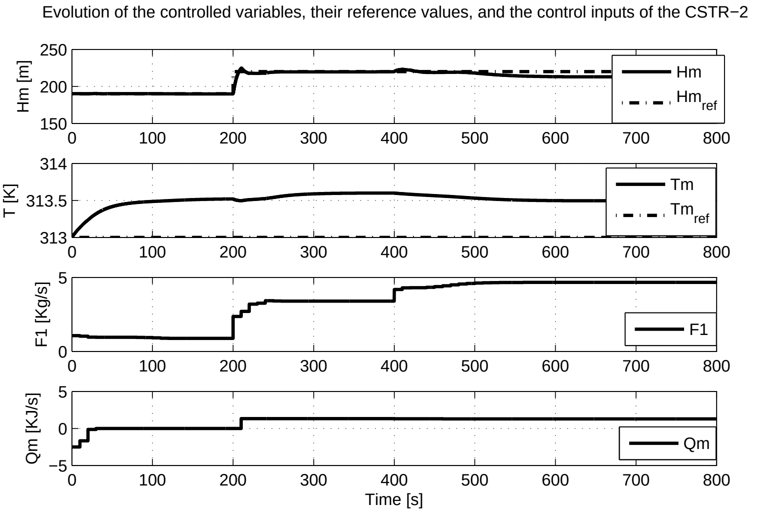

Figure 2.2: Evolution of the level and the temperature of the CSTR-2, their reference values, and the control inputs.

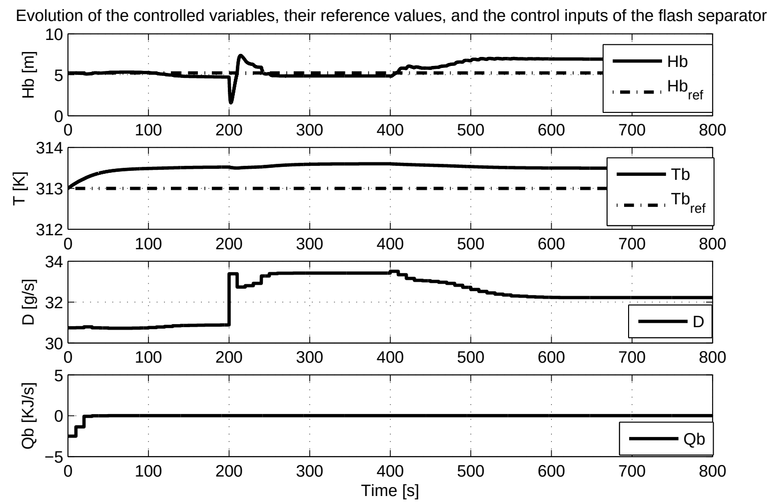

Figure 2.3: Evolution of the level and the temperature of the non-adiabatic flash, their reference values, and the control inputs.

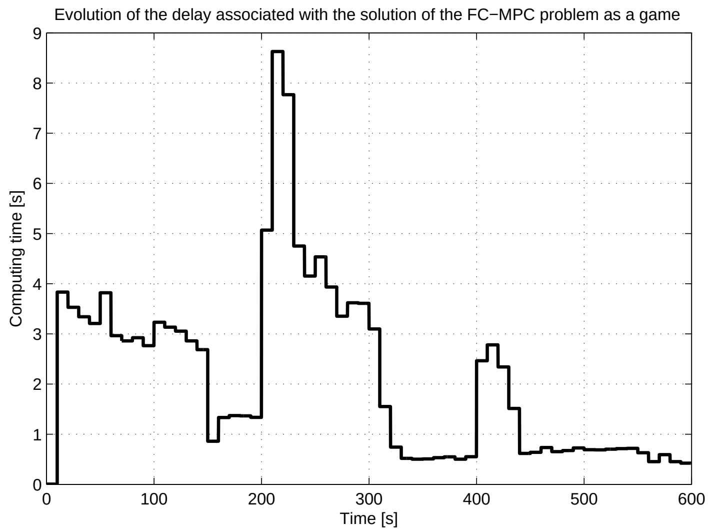

Figure 2.4: Evolution of the delay associated with the solution of the FC-MPC problem as a game.

From Fig. 2.4 it is possible to conclude that the computation of the FC-MPC problem as a game in the case of the chain of reactors presented in this section is always less than the sample time. Moreover, the negotiation model can be stopped prematurely without decreasing the performance of the system. With the purpose of determining the effect of measurement noise in the performance of the proposed control scheme, a measurement noise in the range $[-10,\ 10]$ was added to the controlled variables. The noise signal employed was an uncorrelated white noise signal. In Figs. 2.5 to 2.7 the results of this simulation are presented. In this simulation the prediction horizon $N_\mathrm{p} = 25$, the control horizon $N_\mathrm{u} = 10$, the weight of each subsystem $w = 0.33$, the sample time $T_s = 10s$, and the maximum iterations per time step k, $q_{max} = 5$.
Figures 2.5 to 2.7 show that despite the presence of noise, the local MPC controllers cooperate in order to maintain the performance of each subsystem and the performance of the entire system as well as possible, i.e., the values of the controlled variables of each subsystem as close as possible to their desired values, without affecting negatively the performance of the other subsystems. Such cooperative behavior also involves the decision of some subsystems about not to cooperate. The noncooperative behavior commonly arises in these simulations when a disturbance is applied to the system. In Fig. 2.8, the computation time associated with the computation of the solution of the FC-MPC as a game is presented. Similar to the case without noise, the time employed to compute the optimal control input is less than the sample time.

2.6 Conclusion

In this work, the distributed model predictive control problem was presented. Also the two main approaches to face this problems were briefly introduced: the communication-based model predictive control and the feasible-cooperation model predictive control. For the second approach, a formulation based on concepts of the game theory was carried out. The relevance of this formulation is given by the fact that each subsystem can decide to cooperate or not based on a “rational ”criterion: the benefit that the cooperation gives to each subsystem.

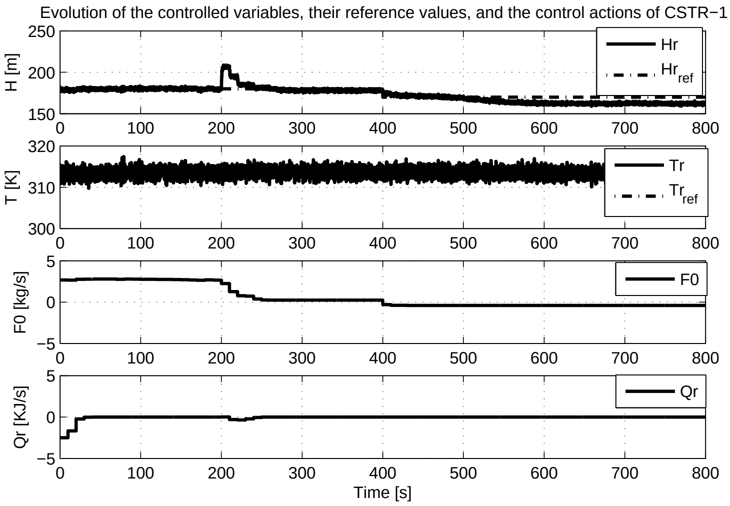

Figure 2.5: Evolution of the level and the temperature of the CSTR-1, their reference values, and the control inputs with a measurement noise

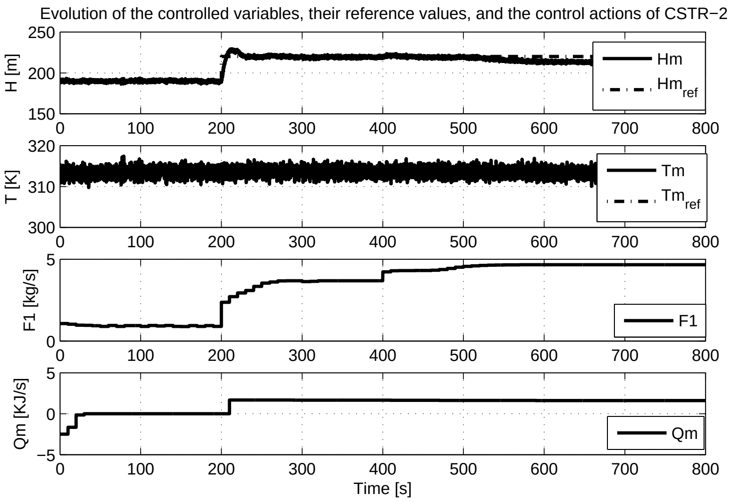

Figure 2.6: Evolution of the level and the temperature of the CSTR-2, their reference values, and the control inputs with a measurement noise

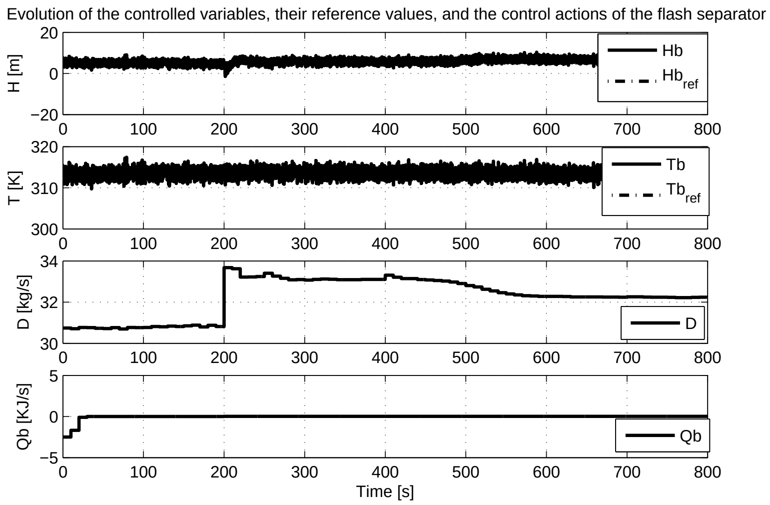

Figure 2.7: Evolution of the level and the temperature of the non-adiabatic flash, their reference values, and the control inputs with a measurement noise

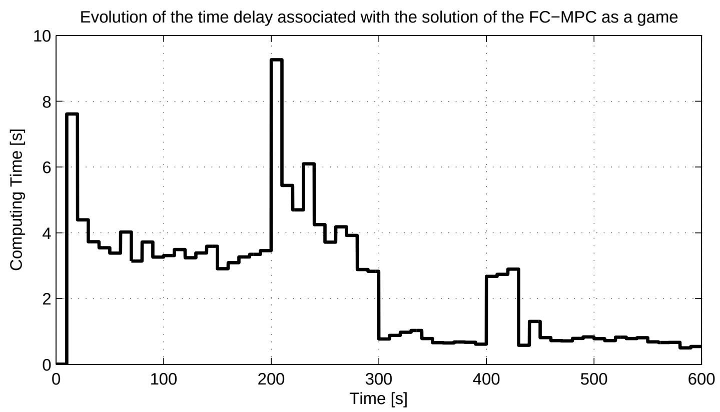

Figure 2.8: Evolution of the delay associated with the solution of the FC-MPC problem as a game.

The proposed control scheme was tested using a chain of two reactors followed by a non-adiabatic flash. The reference values of the reactors was changed in different directions at different times, keeping the values of the references of the other subsystems constant. In this case, the three subsystems cooperate in order to jointly select the best control actions in the sense of the local performance without decreasing the entire system performance. With the purpose of to determinate the effect of the measurement noise in the performance of the proposed control scheme, a measurement noise was added to the controlled variables. Despite of the presence of the noise, the subsystems cooperate in order to maintain the value of the controlled variables close to their reference values.

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