

SEVENTH FRAMEWORK PROGRAMME
THEME – ICT
[Information and Communication Technologies]



Contract Number:	223854
Project Title:	Hierarchical and Distributed Model Predictive Control of Large-Scale Systems
Project Acronym:	HD-MPC



Deliverable Number:	D3.2.3
Deliverable Type:	Report
Contractual Date of Delivery:	September 1, 2011
Actual Date of Delivery:	September 1, 2011
Title of Deliverable:	Reports and publications on the evaluation results, impact on the economics and operability of distributed processes
Dissemination level:	Public
Workpackage contributing to the Deliverable:	WP3
WP Leader:	Wolfgang Marquardt
Partners:	TUD, EDF, POLIMI, RWTH, USE, UNC, SUPELEC
Author(s):	M.A. Ridao and M. Farina

Table of contents

Executive Summary	3
1 Synopsis	4
2 Evaluation Results on the Four-tank System	7
2.1 The four tank plant and proposed experiment	7
2.2 Evaluation results	8
2.2.1 Evaluation of the controllers	8
2.2.2 Evaluation of the experimental results	10
2.3 Conclusions	12
3 An Output Feedback Distributed Predictive Control Algorithm	13
3.1 Introduction	13
3.2 Partitioned systems	13
3.3 The output feedback DPC algorithm	14
3.4 Convergence results	17
3.5 Example	18
3.6 Proof of Theorem 1	19
3.6.1 The collective problem	19
3.6.2 Feasibility	22
3.6.3 Convergence of the optimal cost function	23
3.6.4 Convergence of the trajectories	24
3.7 Conclusions	26
Bibliography	27

Project co-ordinator

Name: Bart De Schutter
Address: Delft Center for Systems and Control
Delft University of Technology
Mekelweg 2, 2628 Delft, The Netherlands
Phone Number: +31-15-2785113
Fax Number: +31-15-2786679
E-mail: b.deschutter@tudelft.nl
Project web site: <http://www.ict-hd-mpc.eu>

Executive Summary

The deliverable starts with a synopsis chapter. The second chapter of this deliverable then describes the evaluation results on a real four-tank system. Seven different approaches were tested and compared, including two centralized MPC, a decentralized MPC and four distributed MPC approaches developed by the HD-MPC Consortium. Some qualitative properties of these controllers are compared and also experimental qualitative results are also shown.

In the third chapter, the state feedback Distributed Predictive Control (henceforth called DPC) algorithm presented in [4] is extended to the output feedback case by the use of standard Luenberger observers for the estimation of the subsystems' states. It is based on a non iterative scheme with neighbor-to-neighbor (i.e., partially connected) communication among the subsystems where partial (local) structural information are needed, and is deeply inspired to the robust state feedback MPC approach first introduced in [8] and subsequently extended to the output feedback problem in [7].

Chapter 1

Synopsis

In order to compare the behavior of the different distributed MPC approaches developed in the scope of the HD-MPC Project a real four-tank plant located in the Department of Ingeniería de Sistemas y Automática of the University of Seville that has been used as a real benchmark system.

Seven different approaches were tested and compared, including two centralized MPC and a decentralized MPC. The tested algorithms were the following:

- Centralized MPC for tracking
- Centralized standard MPC for regulation
- Decentralized MPC for tracking
- Distributed MPC based on a cooperative game
- Sensitivity-Driven Distributed Model Predictive Control
- Feasible-cooperation distributed model predictive controller based on bargaining game theory concepts
- Serial DMPC scheme

The last four ones are distributed MPC algorithms developed by HD-MPC Consortium.

First, some qualitative properties of these controllers are compared. The entry *Model Requirements* shows whether the controllers need full or partial knowledge of the system and whether the model used is linear or nonlinear. The entry *Control Objectives* shows whether the controller is optimal from a centralized point of view (i.e., provides the same solution as the centralized MPC for regulation), guarantees constraint satisfaction if a feasible solution is obtained and whether it can be designed to guarantee closed-loop stability in a regulation problem. The *Auxiliary Software* entry shows which type of additional software is needed by each controller of the distributed scheme.

On the other hand, the experimental qualitative results demonstrate how centralized solutions provide the best performance while the performance of a fully decentralized controller is worse. Distributed schemes in which the controllers communicate in general improve this performance.

The results are summarized in the following table, where J_t is the performance index, J_t the transient performance index, evaluated computing the cumulated cost during the transient. The entry t_s shows the cumulated settling times of the three reference changes.

Qualitative properties	Model Requirements	Control Objectives	Auxiliary Software
Centralized Tracking MPC	Linear system Full model	Suboptimal Constraints Stability	QP
Centralized Regulation MPC	Linear system Full model	Optimal Constraints Stability	QP
Decentralized MPC	Linear system Local model	Suboptimal	QP
DMPC Cooperative game	Linear system Local model (<i>Full model</i>)	Suboptimal Constraints (<i>Stability</i>)	QP
SD-DMPC	Linear system Local model	Optimal Constraints	QP
DMPC Bargaining game	Linear system Local model	Suboptimal Constraints	NLP
Serial DMPC	Linear system Local model	Optimal Constraints	QP

Table 1.1: Table of qualitative properties of each tested controller.

The computational burden is measured on the number and size of the optimization problems solved at each sampling time.

Finally, the communicational burden of each controller is measured by the mean number of floating point numbers that have to be transmitted each sampling time by each agent and the number of communication cycles involved.

In the third chapter, the state feedback Distributed Predictive Control (henceforth called DPC) algorithm presented in [4] is extended to the output feedback case by the use of standard Luenberger observers for the estimation of the subsystems' states. It is based on a non iterative scheme with neighbor-to-neighbor (i.e., partially connected) communication among the subsystems where partial (local) structural information are needed, and is deeply inspired to the robust state feedback MPC approach first introduced in [8] and subsequently extended to the output feedback problem in [7]. The main rationale behind both output-feedback DPC and state-feedback DPC is to transmit among the neighbors the future reference trajectories and to interpret the difference between these trajectories and the true ones as disturbances to be rejected by a proper robust MPC method. Joint constraints between the subsystems could be included, so that a wide range of systems (or systems-of-systems) can be tackled with the present approach. Finally, convergence results can be established.

An off-line design phase must be carried out in order to apply the DPC algorithm:

- 1) Define a decentralized control law (i.e., the auxiliary control law) which, at the same time, (a) stabilizes the local subsystems when neglecting the interconnections, (b) stabilizes the overall large scale system, (c) has a Lyapunov function which basically corresponds to a weighted sum of local Lyapunov functions. The above-mentioned issues can be addressed using a number of well-established results, worked out in the past in the field of decentralized control. For instance, one can rely on milestone results on *connective stability* [9], vector Lyapunov functions and the so-called "weighted sum ap-

Control performance	J	J_t	t_s	N	# floats	# trans
Centralized Tracking MPC	28.4	28.12	3280	5	N.D	N.D.
Centralized Regulation MPC	25.46	23.78	2735	5	N.D	N.D.
Decentralized MPC	39.54	21.2	1685	5	0	0
DMPC Cooperative game	30.71	28.19	2410	5	20	3
SD-DMPC (w/o KF)	35.65	23.28	2505	100	33	10
SD-DMPC (with KF)	28.61	28.26	1895	100	33	10
DMPC Bargaining game	46.32	39.52	3715	5	6	2
Serial DMPC	44.59	41.94	3130	5	10	[2,7]

Table 1.2: Table of the quantitative benchmark indexes of each tested controller

proach” for proving connective stability. More recently, problems (a) and (b) have been successfully addressed in [3], where a small gain condition for large-scale (nonlinear) systems has been derived.

- 2) Define a decentralized Luenberger observer.
- 3) Set the stage and final cost functions.
- 4) Define the proper sets to constrain the state and input trajectories using set-theoretic considerations.
- 5) For each subsystem $i = 1, \dots, M$, define an initial reference state trajectory.

Once the cost functions and the constraining sets are properly defined, the minimization problems to be solved online correspond to low-order MPC problems, defining local subsystem’s inputs. Note that the reference trajectory, for each subsystem, is incrementally defined.

Chapter 2

Evaluation Results on the Four-tank System

2.1 The four tank plant and proposed experiment

The four-tank plant is a laboratory plant located in the Department of Ingeniería de Sistemas y Automática of the University of Seville that has been used as a real benchmark system to test and compare different HD-MPC approaches developed in the scope of the HD-MPC Project. A complete description of the plant and the models can be found in Deliverables D4.3.1 and D4.4.1 and in [2].

A continuous-time state-space model of the quadruple-tank process system can be derived from first principles to result in

$$\begin{aligned}
 \frac{dh_1}{dt} &= -\frac{a_1}{S} \sqrt{2gh_1} + \frac{a_3}{S} \sqrt{2gh_3} + \frac{\gamma_a}{S} q_a, \\
 \frac{dh_2}{dt} &= -\frac{a_2}{S} \sqrt{2gh_2} + \frac{a_4}{S} \sqrt{2gh_4} + \frac{\gamma_b}{S} q_b, \\
 \frac{dh_3}{dt} &= -\frac{a_3}{S} \sqrt{2gh_3} + \frac{(1-\gamma_b)}{S} q_b, \\
 \frac{dh_4}{dt} &= -\frac{a_4}{S} \sqrt{2gh_4} + \frac{(1-\gamma_a)}{S} q_a,
 \end{aligned} \tag{2.1}$$

where h_i , S and a_i with $i \in \{1, 2, 3, 4\}$ refer to the level, cross section and the discharge constant of tank i , respectively; q_j and γ_j with $j \in \{a, b\}$ denote the flow and the ratio of the three-way valve of pump j , respectively and g is the gravitational acceleration.

Seven different approaches were tested and compared, including two centralized MPC and a decentralized MPC. A description of the document can be found in Deliverable D6.5.1. The tested algorithms were the following:

- Centralized MPC for tracking
- Centralized standard MPC for regulation
- Decentralized MPC for tracking
- Distributed MPC based on a cooperative game
- Sensitivity-Driven Distributed Model Predictive Control

- Feasible-cooperation distributed model predictive controller based on bargaining game theory concepts
- Serial DMPC scheme

The last four ones are distributed MPC algorithms developed by HD-MPC Consortium. The description of the algorithm is out of the scope of this deliverable but a short description of them can be found in Deliverable D.5.1 and a complete description in [1]

The four-tank plant is a laboratory plant located in the Department of Ingeniería de Sistemas y Automática of the University of Seville that has been designed to test control techniques using industrial instrumentation and control systems. The plant consists of a hydraulic process of four interconnected tanks inspired by the educational quadruple-tank process proposed by Johansson [6]. A complete description of the plant and the models can be found in Deliverables D4.3.1 and D4.4.1 and in [2]

The following experiment is defined in which the control objective is to follow a set of reference changes in the levels of tanks 1 and 2, h_1 and h_2 , by manipulating the inlet flows q_a and q_b based on the measured levels of the four tanks:

- The first set-points are set to $s_1 = s_2 = 0.65$ m. These are aimed to steer the plant to the operating point and guarantee identical initial conditions for each controller. Once the plant reaches the operating point the benchmark starts maintaining the operation point for 300 seconds.
- In the first step, the set-points are changed to $s_1 = s_2 = 0.3$ m for 3000 seconds.
- Then, the set-points are changed to $s_1 = 0.5$ m and $s_2 = 0.75$ m for 3000 seconds.
- Finally, the set-points are changed to $s_1 = 0.9$ m and $s_2 = 0.75$ m for another 3000 seconds. To perform this change tanks 3 and 4 have to be emptied and filled respectively.

The set-point signals are shown in Figure 2.1. The control test duration is 3 hours and 20 minutes. It is important to remark that the set-points have been chosen in such a way that large changes in the different equilibrium points are involved. This is illustrated in Figure 2.2, where the region of admissible sets points is depicted together with the proposed set-points. Notice that some of them close to the physical limits of the plant in terms of inputs or level of the tanks 3 and 4.

The objective of the benchmark is to design distributed MPC controllers to optimize the performance index

$$J = \sum_{i=0}^{N_{sim}-1} (h_1(i) - s_1(i))^2 + (h_2(i) - s_2(i))^2 + 0.01(q_a(i) - q_a^s(i))^2 + 0.01(q_b(i) - q_b^s(i))^2$$

where q_a^s and q_b^s are the steady manipulable variables of the plant for the set-points s_1 and s_2 calculated from steady conditions of the proposed model of the plant. The tested controllers have been designed using a sampling time of 5 seconds. The performance index measures the response of the plant once it has been steered to the operation point. Then J is calculated during the time period [2700, 12000] seconds, that is, for a total of $N_{sim} = 1860$ samples.

2.2 Evaluation results

2.2.1 Evaluation of the controllers

Table 2.1 shows some qualitative properties of these controllers. The entry *Model Requirements* shows whether the controllers need full or partial knowledge of the system and whether the model

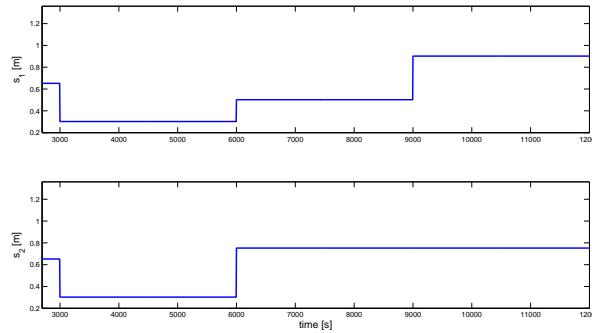


Figure 2.1: Set-point signals for the benchmark

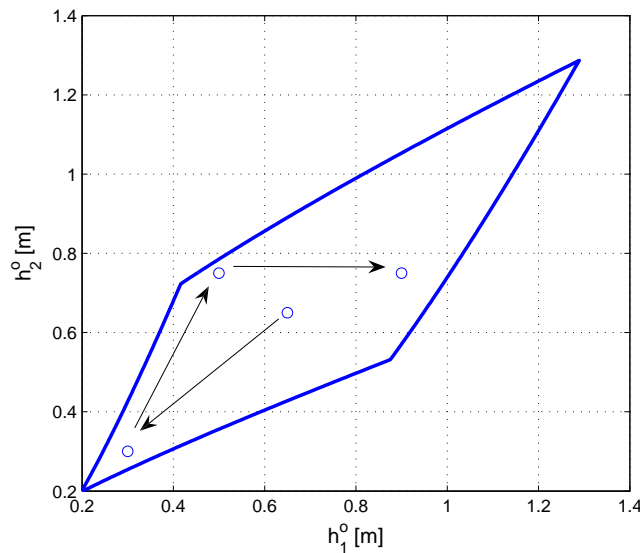


Figure 2.2: Set of admissible set-points.

used is linear or nonlinear. The entry *Control Objectives* shows whether the controller is optimal from a centralized point of view (i.e., provides the same solution as the centralized MPC for regulation), guarantees constraint satisfaction if a feasible solution is obtained and whether it can be designed to guarantee closed-loop stability in a regulation problem. The *Auxiliary Software* entry shows which type of additional software is needed by each controller of the distributed scheme.

The two centralized controllers are based on a linear model of the full plant and are included as a reference for the performance of the distributed MPC schemes. Note that if the controllers could communicate without limits, they would be able to obtain the optimal centralized solution. On the other hand, the decentralized controller provides a reference on what can be achieved with no communication among the controllers at all. All the distributed predictive controllers assume that each agent has access only to its local state and model. All the controllers are based on linear models.

It is worth noting that the centralized MPC for tracking guarantees closed-loop stability not only for regulation problems, but also for tracking problems with any given reference at the cost of optimality. The decentralized controller considered cannot guarantee optimality, constraint satisfaction,

Qualitative properties	Model Requirements	Control Objectives	Auxiliary Software
Centralized Tracking MPC	Linear system Full model	Suboptimal Constraints Stability	QP
Centralized Regulation MPC	Linear system Full model	Optimal Constraints Stability	QP
Decentralized MPC	Linear system Local model	Suboptimal	QP
DMPC Cooperative game	Linear system Local model (<i>Full model</i>)	Suboptimal Constraints (<i>Stability</i>)	QP
SD-DMPC	Linear system Local model	Optimal Constraints	QP
DMPC Bargaining game	Linear system Local model	Suboptimal Constraints	NLP
Serial DMPC	Linear system Local model	Optimal Constraints	QP

Table 2.1: Table of qualitative properties of each tested controller.

nor stability. Note that in order to guarantee closed-loop stability, the DMPC based on a cooperative game needs full model knowledge in order to design the optimization problem of each agent.

The distributed controllers that guarantee optimality (provided sufficient evaluation time) are the Serial DMPC and the SD-DMPC. Note that these controllers are also the ones with a larger communication and computational burden.

Another key issue in distributed schemes is the class of computational capabilities that each controller must have. In particular, for the schemes considered each controller must be able to solve either QP problems or general nonlinear optimization problems. In the experiments, the controllers used MATLAB's optimization toolbox, in particular `quadprog` and `fmincon`.

The properties of each of the proposed controllers are discussed and studied in the previous works which have been included in the references. Note however, that in general, these properties may not hold in the proposed benchmark because the theoretical properties often assume that there are no modeling errors or disturbances and that a given set of assumptions hold. We have carried out all the experiments with the real plant, so there are modeling errors and disturbances. In addition, although most of the controllers are defined for regulation, the benchmark is a reference tracking problem. Issues such as steady state error and disturbance estimation play a relevant role in this benchmark.

2.2.2 Evaluation of the experimental results

The experimental results demonstrate how centralized solutions provide the best performance while the performance of a fully decentralized controller is worse. Distributed schemes in which the controllers communicate in general improve this performance, although the experimental results also demonstrate that a distributed MPC scheme is not necessarily better (according to a certain perfor-

mance index) than a decentralized scheme and it depends on the formulation of the controller and its design.

It is also clear how those controllers that incorporate offset-free techniques (the MPC for tracking, the MPC for regulation and the SD-DMPC with Kalman Filter) provide a better performance index. In order to obtain a measure of the performance without the effect of the steady offset, the transient performance index J_t has been calculated. This index is evaluated computing the cumulated cost during the transient. The entry t_s shows the cumulated settling times of the three reference changes. This shows that those offset-free controllers have a transient performance index similar to the total performance index while for the rest of the controllers, the transient index is better. Note that this index only evaluates the performance during the transient and does not take into account steady state errors. It can be seen that the decentralized scheme shows the shortest settling time t_s and the best transient performance J_t , although this controller exhibits the worst overall performance J . This is due to the fact that the controller reaches a equilibrium point of the controlled system quite fast far from the real set-point.

All the controllers were implemented using a MATLAB function and were not designed to optimize the computational time. For this reason, the computation time has not been taken into account. Computation time has been approximately one second for all controllers. These computation times were lower than the sampling time chosen for each controller and moreover, they could be dramatically reduced using an appropriate implementation framework.

Motivated by these issues, the computational burden is best measured on the number and size of the optimization problems solved at each sampling time. The centralized schemes solve a single QP problem with $2N$ optimization variables while the decentralized controller solves 2 QP problems with N optimization variables. The difference in the computational burden between these schemes grows with the prediction horizon and the number of subsystems. Distributed schemes try to find a trade-off between the burden of computation and communication, and optimality. The DMPC based on a cooperative game and the DMPC based on a bargaining game solve a fixed number of low complexity optimization problems. SD-DMPC and Serial DMPC provide optimality at the cost of a higher computational burden.

On the other hand, the communicational burden of each controller is measured by the mean number of floating point numbers that have to be transmitted each sampling time by each agent and the number of communication cycles involved. It can be seen that iterative DMPC schemes (SD-DMPC and Serial DMPC) in general need to transmit a larger amount of information, while the two controllers based on game theory reach suboptimal cooperative solutions with a lower communicational burden.

The centralized and distributed predictive controllers tested can potentially deal with the satisfaction of hard constraints in the inputs and states of the plant. The experiments demonstrates that all the controllers deal with the limits of the inputs maintaining the feasibility, stability and closed-loop performance. However, the constraints on the states are not active throughout the evolution of the controlled system although there exists states close to the physical limits of the plant. This proves that the tested controllers are capable to take into account the constraints in the calculation of the control action. Besides, the stability, recursive feasibility and constraint satisfaction properties hold in the real experiments, where disturbances and model mismatches between the prediction model and the plant are present.

Control performance	J	J_t	t_s	N	# floats	# trans
Centralized Tracking MPC	28.4	28.12	3280	5	N.D	N.D.
Centralized Regulation MPC	25.46	23.78	2735	5	N.D	N.D.
Decentralized MPC	39.54	21.2	1685	5	0	0
DMPC Cooperative game	30.71	28.19	2410	5	20	3
SD-DMPC (w/o KF)	35.65	23.28	2505	100	33	10
SD-DMPC (with KF)	28.61	28.26	1895	100	33	10
DMPC Bargaining game	46.32	39.52	3715	5	6	2
Serial DMPC	44.59	41.94	3130	5	10	[2,7]

Table 2.2: Table of the quantitative benchmark indexes of each tested controller

2.3 Conclusions

In this chapter, the results of the HD-MPC four-tank benchmark have been presented. In this benchmark, different MPC controllers were applied to the four-tank process plant. These controllers were based on different models and assumptions and provide a broad view of the different distributed MPC schemes developed within the HD-MPC project. The results obtained show how distributed strategies can improve the results obtained by decentralized strategies using the information shared by the controllers.

Chapter 3

An Output Feedback Distributed Predictive Control Algorithm

3.1 Introduction

In this chapter, the state feedback DPC algorithm presented in [4] is extended to the output feedback case by the use of standard Luenberger observers for the estimation of the subsystems' states. It is proven that, under standard assumptions in MPC, the closed-loop system enjoys stability properties, in the sense that the subsystems' state trajectories starting from given sets in the state space converge to the origin. This result is achieved by considering the state estimation error as a further disturbance to be rejected by the control system. Notably, the same considerations developed in the chapter in order to obtain the convergence results can be used to show the robustness of the proposed approach also with respect to exogenous unknown (but bounded) disturbances.

The chapter is organized as follows. In Section 3.2 the partitioned system is introduced, while the output feedback DPC algorithm is defined in Section 3.3. The main convergence results are presented in Section 3.4. Section 3.5 illustrates a simulation example, and some conclusions are drawn in Section 3.7. For clarity of presentation, all the proofs are postponed to the Appendix.

Notation. We say that a matrix is Schur if all its eigenvalues lie in the interior of the unit circle. We use the short-hand $\mathbf{v} = (v_1, \dots, v_s)$ to denote a column vector with s (not necessarily scalar) components v_1, \dots, v_s . The symbol \oplus denotes the Minkowski sum, namely $C = A \oplus B$ if and only if $C = \{c : c = a + b, \text{ for all } a \in A, b \in B\}$. We also denote $\bigoplus_{i=1}^M A_i = A_1 \oplus \dots \oplus A_M$. For a discrete-time signal s_t and $a, b \in \mathbb{N}$, $a \leq b$, we denote $(s_a, s_{a+1}, \dots, s_b)$ with $s_{[a:b]}$. A continuous function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a \mathcal{H}_∞ function iff $\alpha(0) = 0$, it is strictly increasing and $\alpha(s) \rightarrow +\infty$ as $s \rightarrow +\infty$. Finally, $\lambda_M(\cdot)$ and $\lambda_m(\cdot)$ denote the maximum and the minimum eigenvalue of a matrix, respectively.

3.2 Partitioned systems

Consider a large-scale system, which obeys to the linear dynamics

$$\begin{aligned} \mathbf{x}_{t+1} &= \mathbf{A} \mathbf{x}_t + \mathbf{B} \mathbf{u}_t \\ \mathbf{y}_t &= \mathbf{C} \mathbf{x}_t \end{aligned} \tag{3.1}$$

where $\mathbf{x}_t \in \mathbb{R}^n$ is the state vector, and $\mathbf{u}_t \in \mathbb{R}^m$ and $\mathbf{y}_t \in \mathbb{R}^p$ are the input variable and the output variable, respectively.

Let the system (3.1) be partitioned in M low order interconnected non overlapping subsystems, where a generic submodel has $x_t^{[i]} \in \mathbb{R}^{n_i}$ as state vector, i.e., $\mathbf{x}_t = (x_t^{[1]}, \dots, x_t^{[M]})$ and $\sum_{i=1}^M n_i = n$. According to this decomposition, the state transition matrices $A_{11} \in \mathbb{R}^{n_1 \times n_1}, \dots, A_{MM} \in \mathbb{R}^{n_M \times n_M}$ of the M subsystems are diagonal blocks of \mathbf{A} , whereas the non-diagonal blocks of \mathbf{A} (i.e., A_{ij} , with $i \neq j$) define the dynamic coupling between subsystems. Namely, we say that subsystem j is a *dynamic neighbor* of subsystem i if and only if $A_{ij} \neq 0$, i.e. the state of j affects the dynamics of subsystem i . The set of dynamic neighbors of subsystem i (which excludes i) is denoted \mathcal{N}_i .

Furthermore, we assume that the input \mathbf{u}_t and the output \mathbf{y}_t can be partitioned into M input and output vectors $u_t^{[i]} \in \mathbb{R}^{m_i}$ and $y_t^{[i]} \in \mathbb{R}^{p_i}$, respectively, with $i = 1, \dots, M$. We assume that $u_t^{[i]}$ directly affects only the state of the i -th subsystem $x_t^{[i]}$ and $y_t^{[i]}$ only depends on $x_t^{[i]}$, for all $i = 1, \dots, M$. This implies that \mathbf{B} and \mathbf{C} have a block diagonal structure $\mathbf{B} = \text{diag}(B_1, \dots, B_M)$ and $\mathbf{C} = \text{diag}(C_1, \dots, C_M)$, respectively, where $B_i \in \mathbb{R}^{n_i \times m_i}$ and $C_i \in \mathbb{R}^{p_i \times n_i}$ for all $i = 1, \dots, M$. It finally results that the i -th subprocess obeys to the linear dynamics

$$\begin{aligned} x_{t+1}^{[i]} &= A_{ii}x_t^{[i]} + B_i u_t^{[i]} + \sum_{j \in \mathcal{N}_i} A_{ij}x_t^{[j]} \\ y_t^{[i]} &= C_i x_t^{[i]} \end{aligned} \quad (3.2)$$

where we assume that the local states and the local inputs are constrained, i.e., $x_t^{[i]} \in \mathbb{X}_i \subseteq \mathbb{R}^{n_i}$ and $u_t^{[i]} \in \mathbb{U}_i \subseteq \mathbb{R}^{m_i}$, and that the sets \mathbb{X}_i and \mathbb{U}_i are convex neighborhoods of the origin. Furthermore we define $\mathbb{X} = \prod_{i=1}^M \mathbb{X}_i \subseteq \mathbb{R}^n$ and $\mathbb{U} = \prod_{i=1}^M \mathbb{U}_i$, which are convex by convexity of \mathbb{X}_i and \mathbb{U}_i , respectively, for $i = 1, \dots, M$.

We also introduce the collective state constraints, involving more than one subsystem's state

$$H_s(\mathbf{x}_t) \leq 0$$

where $s = 1, \dots, n_c$. We say that H_s is a constraint on subsystem i if $x^{[i]}$ is an argument of H_s . We denote by $\mathcal{C}_i = \{s \in \{1, \dots, n_c\} : H_s \text{ is a constraint on } i\}$ the set of constraints on subsystem i . We say that subsystem j is a *constraint neighbor* of subsystem i if there exists $\bar{s} \in \mathcal{C}_i$ such that $x^{[j]}$ is an argument of $H_{\bar{s}}$, and we let \mathcal{H}_i denote the set of the constraint neighbors of subsystem i . Finally we define, for all $s \in \mathcal{C}_i$, a function $h_s(x^{[i]}, \mathbf{x}) = H_s(\mathbf{x})$, where $x^{[i]}$, the i -th vector component of \mathbf{x} , is not an argument of $H_s(a, \cdot)$. When $\mathbb{X} = \mathbb{R}^n$, $\mathbb{U} = \mathbb{R}^m$ and $n_c = 0$ we say that the system is unconstrained.

The dynamic coupling terms and the coupled constraints induce an interconnected network of subsystems, which can be described by means of a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{I})$, where the nodes in \mathcal{V} are the subsystems and the edge (j, i) in the set $\mathcal{I} \subseteq \mathcal{V} \times \mathcal{V}$ models that the state of j affects the dynamics of subsystem i or j is a constraint neighbor of i . More formally, $(j, i) \in \mathcal{I}$ if and only if $j \in \mathcal{N}_i \cup \mathcal{H}_i$.

3.3 The output feedback DPC algorithm

Our aim is to design, for each subsystem i , an algorithm for computing an input sequence $u_t^{[i]}$ based on the output $y_t^{[i]}$ and some information which is transmitted by its neighbors $\mathcal{N}_i \cup \mathcal{H}_i$, which guarantees closed loop asymptotic convergence to the origin of the state of the large scale system (3.1), the minimization of a given local cost function and constraint satisfaction. Given (3.2), for a given subsystem i we define a local Luenberger observer, which provides an estimate $\bar{x}_t^{[i]}$ of the state $x_t^{[i]}$, based on the local measurement $y_t^{[i]}$, and the state estimates provided by i -th neighbors, i.e., $\bar{x}_t^{[j]}$, $j \in \mathcal{N}_i$. Namely

$$\bar{x}_{t+1}^{[i]} = A_{ii}\bar{x}_t^{[i]} + B_i u_t^{[i]} + \sum_{j \in \mathcal{N}_i} A_{ij}\bar{x}_t^{[j]} - L_i(y_t^{[i]} - C_i\bar{x}_t^{[i]}) \quad (3.3)$$

Assuming that the decentralized estimator (3.3) enjoys the stability properties specified in the following, given the system state initial conditions \mathbf{x}_0 and the observer initial conditions $\bar{\mathbf{x}}_0 = (\bar{x}_0^{[1]}, \dots, \bar{x}_0^{[M]})$, we require that there exist, for all $i = 1, \dots, M$, sets $\Sigma_i \subseteq \mathbb{R}^{n_i}$ such that $\boldsymbol{\sigma}_t = \mathbf{x}_t - \bar{\mathbf{x}}_t \in \Sigma = \prod_{i=1}^M \Sigma_i$ for all $t \geq 0$. This amounts to say that $\boldsymbol{\sigma}_t^{[i]} = x_t^{[i]} - \bar{x}_t^{[i]} \in \Sigma_i$ for all $t \geq 0$, for all $i = 1, \dots, M$.

Furthermore we set, for each subsystem, a reference trajectory $\tilde{x}_t^{[i]}$ which is transmitted to the subsystems which have i as neighbor. We also assume that one can guarantee that, for all $t \geq 0$, the local state estimate $\bar{x}_t^{[i]}$ lies in a specified time-invariant neighborhood of $\tilde{x}_t^{[i]}$ i.e. $\bar{x}_t^{[i]} - \tilde{x}_t^{[i]} \in \mathcal{E}_i$, where $0 \in \mathcal{E}_i$. Note that this, in turn, implies that the real state variable $x_t^{[i]}$ is also guaranteed to lie in a given neighborhood of $\tilde{x}_t^{[i]}$, i.e. $x_t^{[i]} - \tilde{x}_t^{[i]} \in \mathcal{E}_i \oplus \Sigma_i$ for all $i = 1, \dots, M$.

Letting $w_t^{[i]} = \sum_{j \in \mathcal{N}_i} A_{ij}(\bar{x}_t^{[j]} - \tilde{x}_t^{[j]}) - L_i(y_t^{[i]} - C_i \bar{x}_t^{[i]})$, the i -th observer equation (3.3) can be written as follows

$$\bar{x}_{t+1}^{[i]} = A_{ii} \bar{x}_t^{[i]} + B_i u_t^{[i]} + \sum_{j \in \mathcal{N}_i} A_{ij} \tilde{x}_t^{[j]} + w_t^{[i]} \quad (3.4)$$

where the term $w_t^{[i]} \in \mathbb{W}_i = \bigoplus_{j \in \mathcal{N}_i} A_{ij} \mathcal{E}_j \oplus (-L_i C_i) \Sigma_i$ represents a bounded disturbance affecting equation (3.4) and $\sum_{j \in \mathcal{N}_i} A_{ij} \tilde{x}_t^{[j]}$ can be considered as a known input. Provided that, for all $i = 1, \dots, M$, the constraint $\bar{x}_t^{[i]} - \tilde{x}_t^{[i]} \in \mathcal{E}_i$ is satisfied for all $t \geq 0$, we cast the problem of designing an output-feedback distributed controller for the real system as the problem of designing a robust state-feedback control law for the subsystem (3.4), for all $i = 1, \dots, M$.

For the statement of the local MPC sub-problems (i.e., i -DPC problems) we rely on the robust MPC algorithm presented in [8] for constrained linear systems with bounded disturbances, and extended to the output feedback case in [7]. Although this approach requires a more complex off-line design phase with respect to robust MPC methods based on the solution of min-max problems, the optimization problems to be solved on-line are of the same order magnitude than the ones required for non-robust methods.

We define the i -th subsystem nominal model associated to equation (3.4)

$$\hat{x}_{t+1}^{[i]} = A_{ii} \hat{x}_t^{[i]} + B_i \hat{u}_t^{[i]} + \sum_{j \in \mathcal{N}_i} A_{ij} \tilde{x}_t^{[j]} \quad (3.5)$$

The control law, both for the real i -th subsystem (3.2) and for the equation (3.4) will be assigned, for all $t \geq 0$, according to

$$u_t^{[i]} = \hat{u}_t^{[i]} + K_i^{aux} (\bar{x}_t^{[i]} - \hat{x}_t^{[i]}) \quad (3.6)$$

where K_i^{aux} is a suitable control gain. Letting $z_t^{[i]} = \bar{x}_t^{[i]} - \hat{x}_t^{[i]}$ from (3.4) and (3.6) we obtain

$$z_{t+1}^{[i]} = (A_{ii} + B_i K_i^{aux}) z_t^{[i]} + w_t^{[i]} \quad (3.7)$$

where $w_t^{[i]} \in \mathbb{W}_i$. Since \mathbb{W}_i is bounded, if $(A_{ii} + B_i K_i^{aux})$ is Schur, then there exists a robust positively invariant (RPI) set Z_i for (3.7) such that, for all $z_t^{[i]} \in Z_i$, then $z_{t+1}^{[i]} \in Z_i$. From (3.7) it follows that, if $u_k^{[i]}$ is computed as in (3.6) for all $k \geq t$, then

$$\bar{x}_k^{[i]} - \hat{x}_k^{[i]} \in Z_i \quad (3.8)$$

implies that $\bar{x}_k^{[i]} - \hat{x}_k^{[i]} \in Z_i$ for all $k \geq t$.

Now write $\bar{x}_t^{[i]} - \tilde{x}_t^{[i]} = (\bar{x}_t^{[i]} - \hat{x}_t^{[i]}) + (\hat{x}_t^{[i]} - \tilde{x}_t^{[i]})$ and define the set E_i for all $i = 1, \dots, M$ as a set containing the origin and satisfying $E_i \oplus Z_i \subseteq \mathcal{E}_i$. Since, in view of (3.8), $\bar{x}_k^{[i]} - \hat{x}_k^{[i]} \in Z_i$ for all $k \geq t$, if we also satisfy the constraint

$$\hat{x}_k^{[i]} - \tilde{x}_k^{[i]} \in E_i \quad (3.9)$$

for all $k \geq t$, then $\bar{x}_k^{[i]} - \tilde{x}_k^{[i]} \in \mathcal{E}_i$ for all $k \geq t$ as required.

We are now in the position to state the local minimization problem for all subsystems at instant t . Given the future reference trajectories of i and its neighbors $\tilde{x}_k^{[j]}$, $k = t, \dots, t+N-1$, $j \in \mathcal{N}_i \cup \mathcal{H}_i \cup \{i\}$, the i -DPC problem consists in the following

$$\min_{\hat{x}_t^{[i]}, \hat{u}_{[t:t+N-1]}^{[i]}} V_i^N(\hat{x}_t^{[i]}, \hat{u}_{[t:t+N-1]}^{[i]}) \quad (3.10)$$

subject to the dynamic and static constraints (3.5), (3.3), (3.6), (3.8), (3.9), to the local state and input constraints

$$\hat{x}_k^{[i]} \in \hat{\mathbb{X}}_i \quad (3.11)$$

$$\hat{u}_k^{[i]} \in \hat{\mathbb{U}}_i \quad (3.12)$$

where $\hat{\mathbb{X}}_i \oplus Z_i \oplus \Sigma_i \subseteq \mathbb{X}_i$ and $\hat{\mathbb{U}}_i \oplus K_i^{aux} Z_i \subseteq \mathbb{U}_i$ and to the regional state constraints

$$\hat{h}_s^{[i]}(\hat{x}_k^{[i]}, \tilde{\mathbf{x}}_k) \leq 0 \quad (3.13)$$

for $k = t, \dots, t+N-1$, for all $s \in \mathcal{C}_i$, where the function $\hat{h}_s^{[i]}$ is defined in such a way that $\hat{h}_s^{[i]}(\hat{x}_k^{[i]}, \tilde{\mathbf{x}}_k) \leq 0$ guarantees that $h_s(x_k^{[i]}, \mathbf{x}_k^*) \leq 0$ for all $x_k^{[i]} \in \hat{x}_k^{[i]} \oplus Z_i \oplus \Sigma_i$ and $\mathbf{x}_k^* \in \tilde{\mathbf{x}}_k \oplus \prod_{i=1}^M \mathcal{E}_i \oplus \Sigma$. Furthermore, the nominal state trajectory must satisfy the following terminal constraint

$$\hat{x}_{t+N}^{[i]} \in \hat{\mathbb{X}}_i^F \quad (3.14)$$

where $\hat{\mathbb{X}}_i^F$ is the i -th nominal subsystem terminal set, whose properties will be specified in the following.

The cost function $V_i^N(\hat{x}_t^{[i]}, \hat{u}_{[t:t+N-1]}^{[i]})$ is

$$V_i^N(\hat{x}_t^{[i]}, \hat{u}_{[t:t+N-1]}^{[i]}) = \sum_{k=t}^{t+N-1} l_i(\hat{x}_k^{[i]}, \hat{u}_k^{[i]}) + V_i^F(\hat{x}_{t+N}^{[i]}) \quad (3.15)$$

where $l_i : \mathbb{R}^{n_i} \times \mathbb{R}^{m_i} \rightarrow \mathbb{R}_+$ is the stage cost and $V_i^F : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$ is the final cost. From now on, we assume that l_i is defined in such a way that $l_i(0, 0) = 0$ and that there exists, for all $i = 1, \dots, M$, a \mathcal{K}_∞ function α and a matrix R_i satisfying $\text{rank}([B_i^T R_i^T]^T) = m_i$ such that $l_i(\hat{x}^{[i]}, \hat{u}^{[i]}) \geq \alpha(\|(\hat{x}^{[i]}, R_i \hat{u}^{[i]})\|)$ for all $\hat{x}^{[i]} \in \mathbb{R}^{n_i}$, $\hat{u}^{[i]} \in \mathbb{R}^{m_i}$. Note that this assumption can always be fulfilled by a proper choice of the weight R_i in the stage cost.

As in [8], in the stated problem minimization is performed with respect both to the nominal system initial state $\hat{x}_t^{[i]}$ and to the nominal input trajectory $\hat{u}_{[t:t+N-1]}^{[i]}$. Letting the pair $\hat{x}_{t/t}^{[i]}, \hat{u}_{[t:t+N-1]/t}^{[i]}$ be the solution to the i -DPC problem (3.10) at time t , we set the input to the nominal system (3.5), at time t , as $\hat{u}_{t/t}^{[i]}$. According to (3.6), the input to the real system (3.2), at instant t , is

$$u_t^{[i]} = \hat{u}_{t/t}^{[i]} + K_i^{aux}(\bar{x}_t^{[i]} - \hat{x}_{t/t}^{[i]}) \quad (3.16)$$

Furthermore, let us define as $\hat{x}_{k/t}^{[i]}$ the trajectory stemming from $\hat{x}_{t/t}^{[i]}$ and $\hat{u}_{[t:t+N-1]/t}^{[i]}$, in view of equation (3.5). The value of the reference state variable $\tilde{x}_{t+N}^{[i]}$ is set to

$$\tilde{x}_{t+N}^{[i]} = \hat{x}_{t+N/t}^{[i]} \quad (3.17)$$

We stress that we do not define, at each instant t , a new reference trajectory $\tilde{x}_k^{[i]}$, $k = t + 1, \dots, t + N$, but we append the value $\tilde{x}_{t+N}^{[i]}$ to the reference trajectory which has been already defined for $k \leq t + N - 1$.

3.4 Convergence results

The following definitions and assumptions are needed to state the main result of the chapter. The sets of admissible initial conditions \mathbf{x}_0 , $\bar{\mathbf{x}}_0$, and $\tilde{\mathbf{x}}_{[0:N-1]}^{[j]}$, for all $j = 1 \dots, M$ are defined as follows.

Definition 1 Letting $\mathbf{x} = (x^{[1]}, \dots, x^{[M]})$, we denote the feasibility region \mathbb{X}^N for all the i -DPC problems as the set

$$\begin{aligned} \mathbb{X}^N := & \{ \mathbf{x} : \text{if } x_0^{[i]} = x^{[i]} \text{ for all } i = 1, \dots, M \\ & \text{then } \exists \bar{\mathbf{x}}_0, (\tilde{\mathbf{x}}_{[0:N-1]}^{[1]}, \dots, \tilde{\mathbf{x}}_{[0:N-1]}^{[M]}), (\hat{x}_{0/0}^{[1]}, \dots, \hat{x}_{0/0}^{[M]}), \\ & (\hat{u}_{[0:N-1]}^{[1]}, \dots, \hat{u}_{[0:N-1]}^{[M]}) \text{ such that (3.2), (3.8), (3.9),} \\ & \text{(3.11)-(3.14) are satisfied for all } i = 1, \dots, M \} \end{aligned}$$

We denote, for each $\mathbf{x} \in \mathbb{X}^N$, the region of feasible initial state estimates. Letting $\bar{\mathbf{x}} = (\bar{x}^{[1]}, \dots, \bar{x}^{[M]})$

$$\begin{aligned} \bar{\mathbb{X}}_{\mathbf{x}}^N := & \{ \bar{\mathbf{x}} : \text{if } x_0^{[i]} = x^{[i]} \text{ and } \bar{x}_0^{[i]} = \bar{x}^i \text{ for all } i = 1, \dots, M \\ & \text{then } \exists (\tilde{\mathbf{x}}_{[0:N-1]}^{[1]}, \dots, \tilde{\mathbf{x}}_{[0:N-1]}^{[M]}), (\hat{x}_{0/0}^{[1]}, \dots, \hat{x}_{0/0}^{[M]}), \\ & (\hat{u}_{[0:N-1]}^{[1]}, \dots, \hat{u}_{[0:N-1]}^{[M]}) \text{ such that (3.2), (3.8), (3.9),} \\ & \text{(3.11)-(3.14) are satisfied for all } i = 1, \dots, M \} \end{aligned}$$

Also, given $\mathbf{x} \in \mathbb{X}^N$ and $\bar{\mathbf{x}} \in \bar{\mathbb{X}}_{\mathbf{x}}^N$, the region of feasible initial reference trajectories is

$$\begin{aligned} \tilde{\mathbb{X}}_{\mathbf{x}, \bar{\mathbf{x}}}^N := & \{ (\tilde{\mathbf{x}}_{[0:N-1]}^{[1]}, \dots, \tilde{\mathbf{x}}_{[0:N-1]}^{[M]}) : \text{if } x_0^{[i]} = x^i \text{ and } \bar{x}_0^{[i]} = \bar{x}^i \\ & \text{for all } i = 1, \dots, M \text{ then } \exists (\hat{x}_{0/0}^{[1]}, \dots, \hat{x}_{0/0}^{[M]}), \\ & (\hat{u}_{[0:N-1]}^{[1]}, \dots, \hat{u}_{[0:N-1]}^{[M]}) \text{ such that (3.2), (3.8), (3.9),} \\ & \text{(3.11)-(3.14) are satisfied for all } i = 1, \dots, M \} \end{aligned}$$

Assumption 1 Letting $\mathbf{L} = \text{diag}(L_1, \dots, L_M)$, the matrix $\mathbf{A} + \mathbf{LC}$ is Schur. Furthermore, there exist, for all $i = 1, \dots, M$, sets $\Sigma_i \subset \mathbb{R}^{n_i}$ such that Σ is a positively invariant set for the system $\sigma_{t+1} = (\mathbf{A} + \mathbf{LC})\sigma_t$.

Assumption 2 The matrix $A_{ii} + B_i K_i^{\text{aux}}$ is Schur, for all $i = 1, \dots, M$.

Assumption 3 Letting $\mathbf{K}^{\text{aux}} = \text{diag}(K_1^{\text{aux}}, \dots, K_M^{\text{aux}})$, $\hat{\mathbf{X}} = \prod_{i=1}^M \hat{\mathbf{X}}_i$, $\hat{\mathbf{U}} = \prod_{i=1}^M \hat{\mathbf{U}}_i$ and $\hat{\mathbf{X}}^F = \prod_{i=1}^M \hat{\mathbf{X}}_i^F$, it holds that:

- (i) The matrix $\mathbf{A} + \mathbf{BK}^{\text{aux}}$ is Schur;

- (ii) $\hat{H}_s^{[i]}(\hat{\mathbf{x}}) \leq 0$ for all $\hat{\mathbf{x}} \in \hat{\mathbb{X}}^F$, for all $s \in \mathcal{C}_i$, for all $i = 1, \dots, M$, where $\hat{H}^{[i]}$ is defined in such a way that $\hat{H}_s^{[i]}(\hat{\mathbf{x}}) = \hat{h}_s^{[i]}(\hat{x}^{[i]}, \hat{\mathbf{x}})$ for all $s \in \mathcal{C}_i$, for all $i = 1, \dots, M$.
- (iii) $\hat{\mathbb{X}}^F \subseteq \hat{\mathbb{X}}$ is an invariant set for $\hat{\mathbf{x}}^+ = (\mathbf{A} + \mathbf{BK}^{aux})\hat{\mathbf{x}}$;
- (iv) $\hat{\mathbf{u}} = \mathbf{K}^{aux}\hat{\mathbf{x}} \in \hat{\mathbb{U}}$ for any $\hat{\mathbf{x}} \in \hat{\mathbb{X}}^F$;
- (v) for all $\hat{\mathbf{x}} \in \hat{\mathbb{X}}^F$ and, for a given constant $\kappa > 0$

$$\mathbf{V}^F(\hat{\mathbf{x}}^+) - \mathbf{V}^F(\hat{\mathbf{x}}) \leq -(1 + \kappa)\mathbf{l}(\hat{\mathbf{x}}, \hat{\mathbf{u}}) \quad (3.18)$$

where

$$\mathbf{V}^F(\hat{\mathbf{x}}) = \sum_{i=1}^M V_i^F(\hat{x}^{[i]}) \quad \text{and}$$

$$\mathbf{l}(\hat{\mathbf{x}}, \hat{\mathbf{u}}) = \sum_{i=1}^M l_i(\hat{x}^{[i]}, \hat{u}^{[i]}).$$

Assumption 4 Given the sets \mathcal{E}_i and the RPI sets Z_i for equations (3.7), there exists a real positive constant $\bar{\rho}_E > 0$ such that $Z_i \oplus \mathcal{B}_{\bar{\rho}_E}(0) \subseteq \mathcal{E}_i$ for all $i = 1, \dots, M$, where $\mathcal{B}_{\bar{\rho}_E}(0)$ is a ball of radius $\bar{\rho}_E > 0$ centered at the origin.

Proper ways to select the design parameters fulfilling Assumptions 2-4 are discussed in the paper [5], where arguments similar to the ones introduced in [9] and [3] are employed. Now we are in the position to state the main result.

Theorem 1 Let Assumptions 1-4 be satisfied and let E_i be a neighborhood of the origin satisfying $E_i \oplus Z_i \subseteq \mathcal{E}_i$. Then, for any initial reference trajectory in $\tilde{\mathbb{X}}_{\mathbf{x}_0, \bar{\mathbf{x}}_0}^N$, the trajectory \mathbf{x}_t , starting from any initial condition $\mathbf{x}_0 \in \mathbb{X}^N$, $\bar{\mathbf{x}}_0 \in \tilde{\mathbb{X}}_{\mathbf{x}_0}^N$, asymptotically converges to the origin.

3.5 Example

Consider the example illustrated in Figure 3.1 consisting in four trucks with masses $m_1 = 3$, $m_2 = 2$, $m_3 = 3$, $m_4 = 6$, each endowed with an individual engine (exerting the force $100u_t^{[i]}$, $i = 1, \dots, 4$). Trucks 1 and 2 (respectively 3 and 4) are dynamically coupled through a spring and a damper, whose coefficients are $k_{12} = 0.5$ and $h_{12} = 0.2$ ($k_{34} = 1$ and $h_{34} = 0.3$), respectively. The components of the

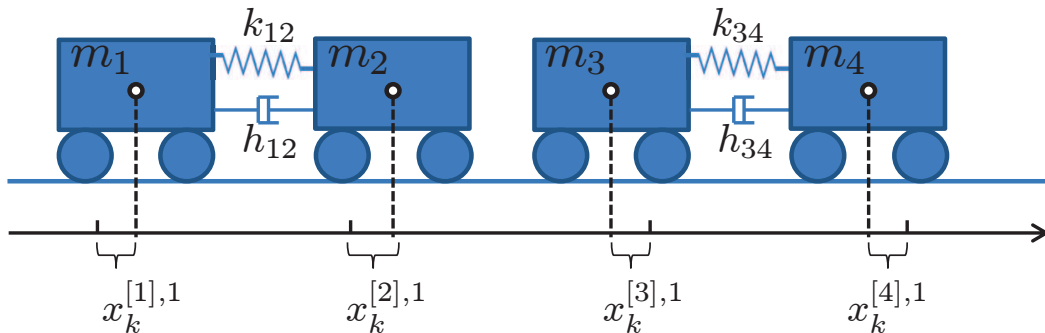


Figure 3.1: Illustration of the example.

2-dimensional state vector $x_t^{[i]}$ of the i -th truck represent the displacement of i with respect to a given equilibrium position (i.e., $x_k^{[i],1}$) and the absolute velocity of the truck. For all $i = 1, \dots, 4$, positions

are measured, i.e., $y_k^{[i]} = x_k^{[i],1}$. The following constraints are set to the input signals: $|u_k^{[i]}| \leq 0.5$ for $i = 1, \dots, 3$ and $|u_k^{[4]}| \leq 1$. The model is discretized with sampling interval $\tau = 0.1$ s. We set the observer's initial conditions to $\bar{x}_0^{[i]} = [5, 0]^T$ and the real system initial conditions are randomly generated in such a way that $x_0^{[i]} - \bar{x}_0^{[i]} \in \Sigma_i$, where Σ_i , satisfying Assumption 1 (L_i are defined by pole assignment, where the poles of the local systems are 0.5 and 0.6 for all i), are shown in Figure 3.2.

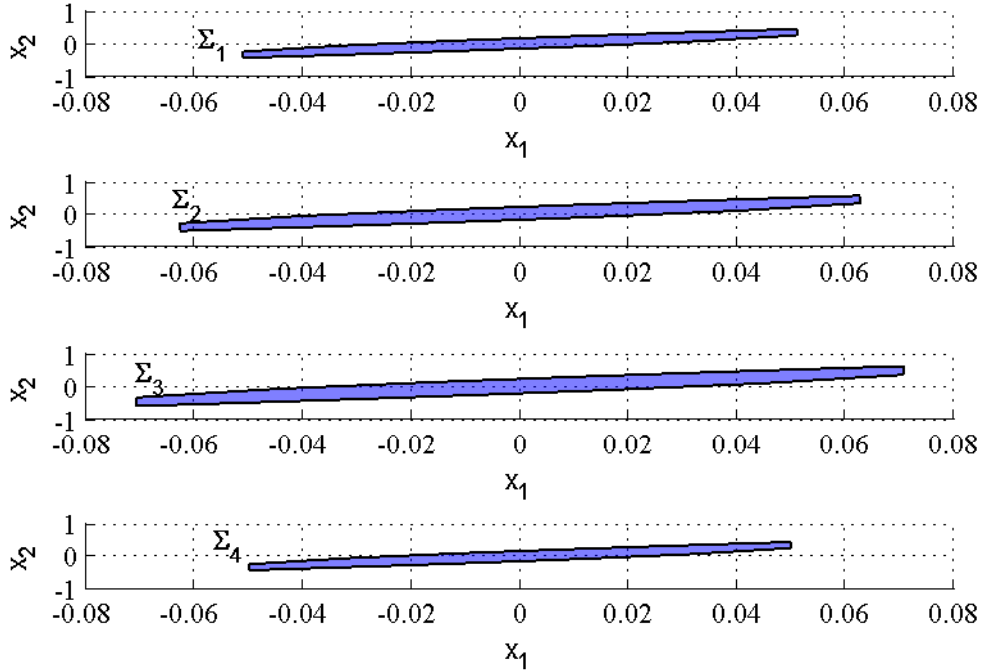


Figure 3.2: Sets Σ_i , $i=1, \dots, 4$.

We define the decentralized control law by pole assignment (the poles of the local systems are 0.5 and 0.6 for all i).

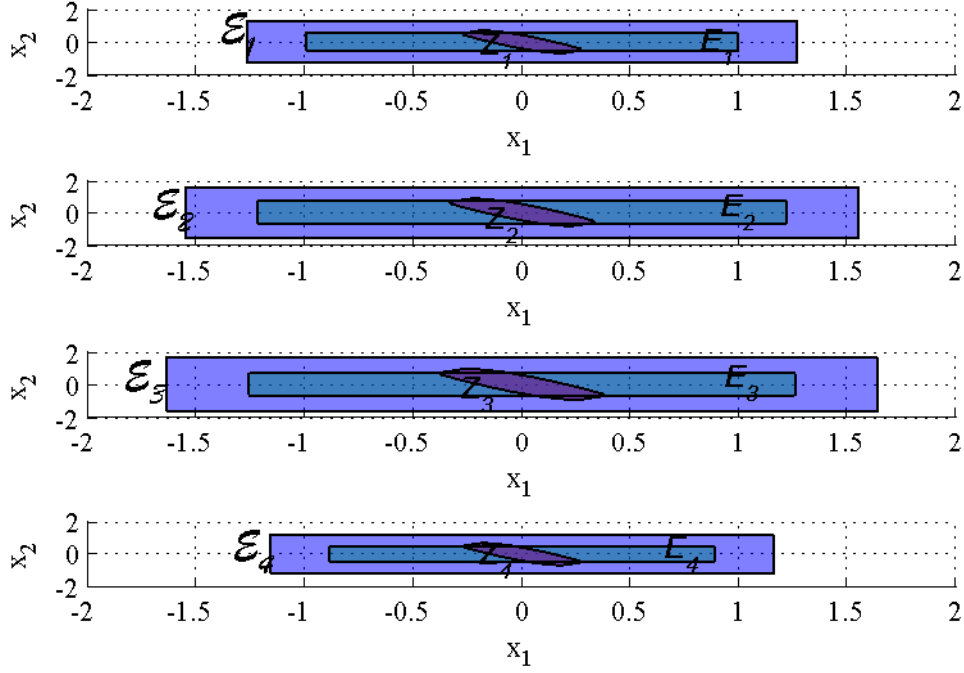
We properly define quadratic weighting functions and we set sets \mathcal{E}_i , Z_i and E_i as in Figure 3.3 (for details see [5]).

The initial position reference trajectories are exponential, starting from the initial conditions, with decaying rate 0.96. The velocity reference trajectories are computed coherently with the position reference trajectories, and $N = 30$. In Fig. 3.4 the plots of the optimal input trajectories obtained with DPC are shown, and in Fig. 3.5 we show the obtained optimal trajectories of the state.

3.6 Proof of Theorem 1

3.6.1 The collective problem

Define the collective vectors $\hat{\mathbf{x}}_t = (\hat{x}_t^{[1]}, \dots, \hat{x}_t^{[M]})$, $\tilde{\mathbf{x}}_t = (\tilde{x}_t^{[1]}, \dots, \tilde{x}_t^{[M]})$, $\hat{\mathbf{u}}_t = (\hat{u}_t^{[1]}, \dots, \hat{u}_t^{[M]})$, $\mathbf{w}_t = (w_t^{[1]}, \dots, w_t^{[M]})$ and $\mathbf{z}_t = (z_t^{[1]}, \dots, z_t^{[M]})$. Furthermore, de-

Figure 3.3: Sets $\mathcal{E}_i, Z_i, E_i, i = 1, \dots, 4$.

fine the matrices $\mathbf{A}^* = \text{diag}(A_{11}, \dots, A_{MM})$, $\tilde{\mathbf{A}} = \mathbf{A} - \mathbf{A}^*$. Collectively, we write equations (3.3), (3.4), and (3.5) as

$$\bar{\mathbf{x}}_{t+1} = \mathbf{A}\bar{\mathbf{x}}_t + \mathbf{B}\mathbf{u}_t - \mathbf{L}(\mathbf{y}_t - \mathbf{C}\bar{\mathbf{x}}_t) \quad (3.19)$$

$$\bar{\mathbf{x}}_{t+1} = \mathbf{A}^*\bar{\mathbf{x}}_t + \mathbf{B}\mathbf{u}_t + \tilde{\mathbf{A}}\bar{\mathbf{x}}_t + \mathbf{w}_t \quad (3.20)$$

$$\hat{\mathbf{x}}_{t+1} = \mathbf{A}^*\hat{\mathbf{x}}_t + \mathbf{B}\hat{\mathbf{u}}_t + \tilde{\mathbf{A}}\hat{\mathbf{x}}_t \quad (3.21)$$

In view of (3.6)

$$\mathbf{u}_t = \hat{\mathbf{u}}_t + \mathbf{K}^{aux}(\bar{\mathbf{x}}_t - \hat{\mathbf{x}}_t) \quad (3.22)$$

and we collectively write (3.7) as

$$\mathbf{z}_{t+1} = (\mathbf{A}^* + \mathbf{B}\mathbf{K}^{aux})\mathbf{z}_t + \mathbf{w}_t \quad (3.23)$$

Since each i -DPC problem depends upon local variables (the coupling terms $\tilde{x}_k^{[i]}$ are fixed for $k = t, \dots, t+N-1$), minimizing (3.10) for all $i = 1, \dots, M$ is equivalent to minimize

$$\mathbf{V}^{N*}(\bar{\mathbf{x}}_t) = \min_{\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_{[t:t+N-1]}} \mathbf{V}^N(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_{[t:t+N-1]}) \quad (3.24)$$

subject to the dynamic constraints (3.21), (3.19), (3.22), the static constraints

$$\bar{\mathbf{x}}_t - \hat{\mathbf{x}}_t \in \mathbb{Z} = \prod_{i=1}^M Z_i \quad (3.25a)$$

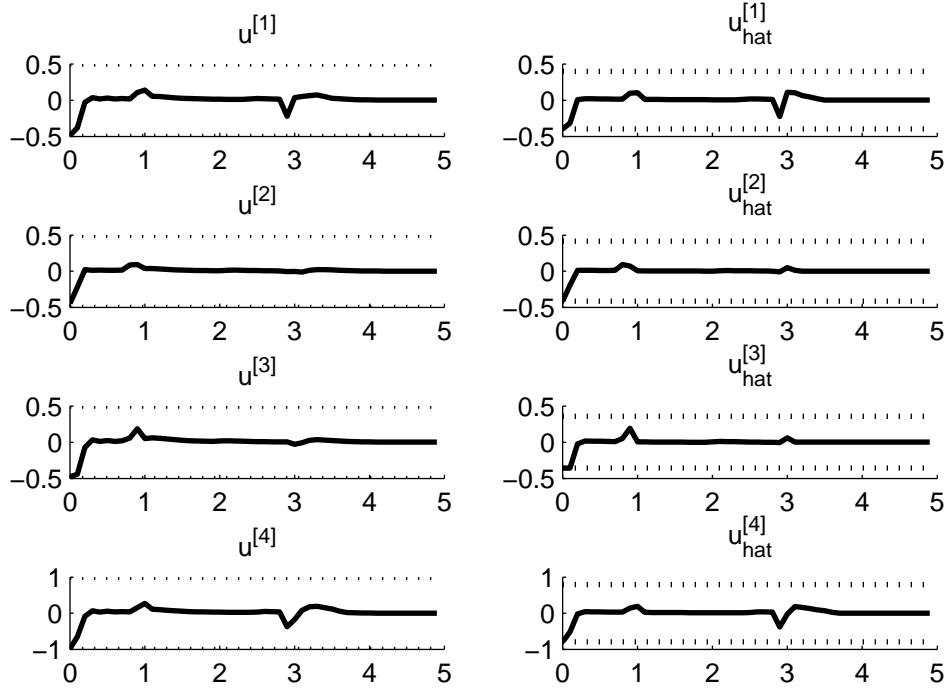


Figure 3.4: Input trajectories. Left: $u_t^{[i]}$ (solid line), thresholds for $u_t^{[i]}$ (dotted lines). Right: $\hat{u}_t^{[i]}$ (solid line), thresholds for $\hat{u}_t^{[i]}$ (dotted lines).

$$\hat{\mathbf{x}}_k - \tilde{\mathbf{x}}_k \in \mathbb{E} = \prod_{i=1}^M E_i \quad (3.25b)$$

$$\hat{\mathbf{x}}_k \in \hat{\mathbb{X}} \quad (3.25c)$$

$$\hat{\mathbf{u}}_k \in \hat{\mathbb{U}} \quad (3.25d)$$

$$\mathbf{H}(\hat{\mathbf{x}}_k, \tilde{\mathbf{x}}_k) \leq 0 \quad (3.25e)$$

for $k = t, \dots, t + N - 1$, and the terminal constraint

$$\hat{\mathbf{x}}_{t+N} \in \hat{\mathbb{X}}^F \quad (3.26)$$

In (3.25), \mathbf{H} collects all the constraints (3.13) and note that, by (ii) in Assumption 3, $\mathbf{H}(\hat{\mathbf{x}}, \hat{\mathbf{x}}) \leq 0$ for all $\hat{\mathbf{x}} \in \hat{\mathbb{X}}^F$. The collective cost function \mathbf{V}^N is defined as

$$\mathbf{V}^N(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_{[t:t+N-1]}) = \sum_{k=t}^{t+N-1} \mathbf{l}(\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_k) + \mathbf{V}^F(\hat{\mathbf{x}}_{t+N})$$

We also define

$$\mathbf{V}^{N,0}(\hat{\mathbf{x}}_t) = \min_{\hat{\mathbf{u}}_{[t:t+N-1]}} \mathbf{V}^N(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_{[t:t+N-1]}) \quad (3.27)$$

subject to the dynamic constraints (3.21) and the static constraints (3.25b)-(3.26).

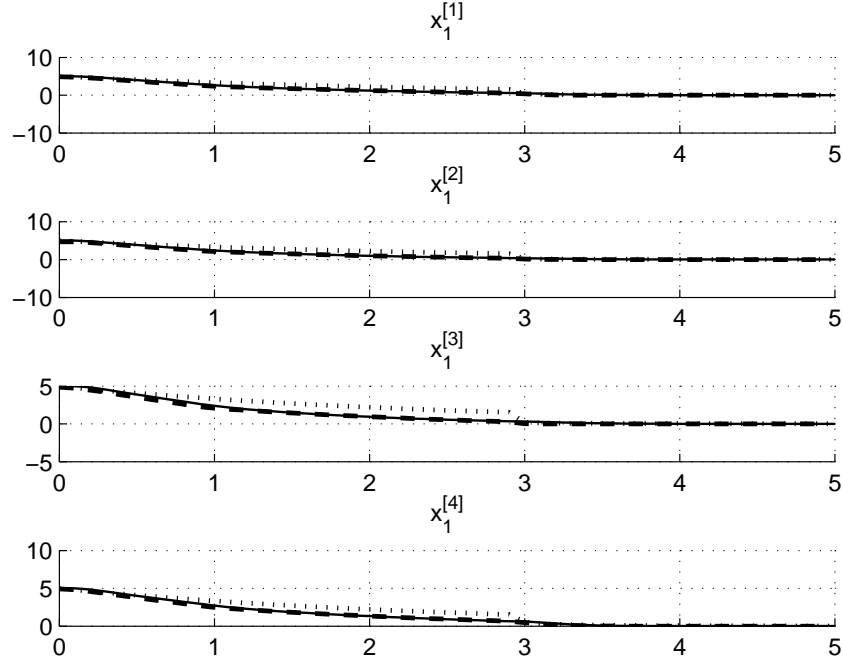


Figure 3.5: Controlled state variables with DPC (solid lines) first entries of $\tilde{x}_t^{[i]}$ (dotted lines) and of $\hat{x}_t^{[i]}$ (dashed lines).

3.6.2 Feasibility

From Definition 1, it collectively holds that

$$\mathbb{X}^N = \{ \mathbf{x} : \text{if } \mathbf{x}_0 = \mathbf{x} \text{ then } \exists \bar{\mathbf{x}}_0, \tilde{\mathbf{x}}_{[0:N-1]}, \hat{\mathbf{x}}_{0/0}, \hat{\mathbf{u}}_{[0:N-1]} \text{ such that (3.21), (3.25) and (3.26) are satisfied} \}$$

For each point of the feasibility set $\mathbf{x} \in \mathbb{X}^N$

$$\tilde{\mathbb{X}}_{\mathbf{x}}^N := \{ \bar{\mathbf{x}}_0 : \text{if } \mathbf{x}_0 = \mathbf{x} \text{ and } \bar{\mathbf{x}}_0 = \bar{\mathbf{x}} \text{ then } \exists \tilde{\mathbf{x}}_{[0:N-1]}, \hat{\mathbf{x}}_{0/0}, \hat{\mathbf{u}}_{[0:N-1]} \text{ such that (3.21), (3.25) and (3.26) are satisfied} \}$$

Finally, if $\mathbf{x} \in \mathbb{X}^N$, $\bar{\mathbf{x}} \in \tilde{\mathbb{X}}_{\mathbf{x}}^N$

$$\tilde{\tilde{\mathbb{X}}}_{\mathbf{x}, \bar{\mathbf{x}}}^N := \{ \tilde{\mathbf{x}}_{[0:N-1]} : \text{if } \mathbf{x}_0 = \mathbf{x} \text{ and } \bar{\mathbf{x}}_0 = \bar{\mathbf{x}} \text{ then } \exists \hat{\mathbf{x}}_{0/0}, \hat{\mathbf{u}}_{[0:N-1]} \text{ such that (3.21), (3.25) and (3.26) are satisfied} \}$$

Assume that, at instant t , $\mathbf{x}_t \in \mathbb{X}^N$, $\bar{\mathbf{x}}_t \in \tilde{\mathbb{X}}_{\mathbf{x}_t}^N$, and that $\tilde{\mathbf{x}}_{[t:t+N-1]} \in \tilde{\tilde{\mathbb{X}}}_{\mathbf{x}_t, \bar{\mathbf{x}}_t}^N$. The optimal nominal input and state sequences obtained by minimizing the collective MPC problem are $\hat{\mathbf{u}}_{[t:t+N-1]/t} = \{\hat{\mathbf{u}}_{t/t}, \dots, \hat{\mathbf{u}}_{t+N-1/t}\}$ and $\hat{\mathbf{x}}_{[t:t+N]/t} = \{\hat{\mathbf{x}}_{t/t}, \dots, \hat{\mathbf{x}}_{t+N/t}\}$, respectively. Finally, recall that it is set $\tilde{\mathbf{x}}_{t+N} = \hat{\mathbf{x}}_{t+N/t}$.

Denote $\hat{\mathbf{u}}_{t+N/t} = \mathbf{K}^{aux} \hat{\mathbf{x}}_{t+N/t}$ and $\hat{\mathbf{x}}_{t+N+1/t} = \mathbf{A}^* \hat{\mathbf{x}}_{t+N/t} + \mathbf{B} \hat{\mathbf{u}}_{t+N/t} + \tilde{\mathbf{A}} \tilde{\mathbf{x}}_{t+N}$. Since $\tilde{\mathbf{x}}_{t+N} = \hat{\mathbf{x}}_{t+N/t}$, the latter is equivalent to $\hat{\mathbf{x}}_{t+N+1/t} = (\mathbf{A} + \mathbf{B} \mathbf{K}^{aux}) \hat{\mathbf{x}}_{t+N/t}$.

Note that, in view of constraint (3.26) and Assumption 3, $\hat{\mathbf{u}}_{t+N/t} \in \hat{\mathcal{U}}$ and $\hat{\mathbf{x}}_{t+N+1/t} \in \hat{\mathcal{X}}^F$. Therefore, they satisfy constraints (3.25c), (3.25d) and (3.26). Also, according to Assumption 3, (3.18) holds.

We also define the input sequence

$$\hat{\mathbf{u}}_{[t+1:t+N]/t} = \{\hat{\mathbf{u}}_{t+1/t}, \dots, \hat{\mathbf{u}}_{t+N-1/t}, \hat{\mathbf{u}}_{t+N/t}\}$$

and the state sequence stemming from the initial condition $\hat{\mathbf{x}}_{t+1/t}$ and the input sequence $\hat{\mathbf{u}}_{[t+1:t+N]/t}$ i.e.,

$$\hat{\mathbf{x}}_{[t+1:t+N+1]/t} = \{\hat{\mathbf{x}}_{t+1/t}, \dots, \hat{\mathbf{x}}_{t+N/t}, \hat{\mathbf{x}}_{t+N+1/t}\}$$

Notice that $\mathbf{x}_k - \bar{\mathbf{x}}_k \in \Sigma$ for all $k = t, \dots, t+N-1$ from Assumption 1, and that, in view of (3.25a)-(3.25b), $\mathbf{w}_k \in \prod_{i=1}^M \mathbb{W}_i$ for all $k = t, \dots, t+N-1$. In view of the feasibility of the i -DPC problem at time t , we have that $\bar{\mathbf{x}}_{t+1} - \hat{\mathbf{x}}_{t+1/t} \in \mathbb{Z}$ and $\hat{\mathbf{x}}_{k/t} - \bar{\mathbf{x}}_k \in \prod_{i=1}^M \mathbb{E}$ for all $k = t+1, \dots, t+N-1$. Note also that $\hat{\mathbf{x}}_{t+N/t} - \bar{\mathbf{x}}_{t+N} = 0 \in \mathbb{E}$ by (3.17). Furthermore, since $\bar{\mathbf{x}}_{t+N} = \hat{\mathbf{x}}_{t+N/t}$ and $\hat{\mathbf{x}}_{t+N} \in \hat{\mathcal{X}}^F$, from (3.26) it holds that $\mathbf{H}(\hat{\mathbf{x}}_{t+N/t}, \bar{\mathbf{x}}_{t+N}) \leq 0$ from (ii) of Assumption 3. Therefore, we can conclude that the state and the input sequences $\hat{\mathbf{x}}_{[t+1:t+N+1]/t}$ and $\hat{\mathbf{u}}_{[t+1:t+N]/t}$ are feasible at $t+1$, since constraints (3.25) and (3.26) are satisfied. This proves that $\mathbf{x}_t \in \mathbb{X}^N$, $\bar{\mathbf{x}}_t \in \tilde{\mathbb{X}}_{\mathbf{x}_t}^N$ and $\tilde{\mathbf{x}}_{[t:t+N-1]} \in \tilde{\mathbb{X}}_{\mathbf{x}_t, \bar{\mathbf{x}}_t}^N$ implies that $\mathbf{x}_{t+1} \in \mathbb{X}^N$, $\bar{\mathbf{x}}_{t+1} \in \tilde{\mathbb{X}}_{\mathbf{x}_{t+1}}^N$ and $\tilde{\mathbf{x}}_{[t+1:t+N]} \in \tilde{\mathbb{X}}_{\mathbf{x}_{t+1}, \bar{\mathbf{x}}_{t+1}}^N$.

3.6.3 Convergence of the optimal cost function

By optimality, $\mathbf{V}^{N,0}(\hat{\mathbf{x}}_{t+1/t}) \leq \mathbf{V}^N(\hat{\mathbf{x}}_{t+1/t}, \hat{\mathbf{u}}_{[t+1:t+N]/t})$, where

$$\mathbf{V}^N(\hat{\mathbf{x}}_{t+1/t}, \hat{\mathbf{u}}_{[t+1:t+N]/t}) = \sum_{k=t+1}^{t+N} \mathbf{l}(\hat{\mathbf{x}}_{k/t}, \hat{\mathbf{u}}_{k/t}) + \mathbf{V}^F(\hat{\mathbf{x}}_{t+N+1/t}) \quad (3.28)$$

Therefore we compute that

$$\begin{aligned} \mathbf{V}^{N,0}(\hat{\mathbf{x}}_{t+1/t}) - \mathbf{V}^{N,0}(\hat{\mathbf{x}}_{t/t}) &\leq -\mathbf{l}(\hat{\mathbf{x}}_{t/t}, \hat{\mathbf{u}}_{t/t}) + \mathbf{l}(\hat{\mathbf{x}}_{t+N/t}, \hat{\mathbf{u}}_{t+N/t}) + \\ &\quad + \mathbf{V}^F(\hat{\mathbf{x}}_{t+N+1/t}) - \mathbf{V}^F(\hat{\mathbf{x}}_{t+N/t}) \end{aligned} \quad (3.29)$$

In view of (3.18)

$$\begin{aligned} \mathbf{V}^F(\hat{\mathbf{x}}_{t+N+1/t}) &\quad - \mathbf{V}^F(\hat{\mathbf{x}}_{t+N/t}) + \mathbf{l}(\hat{\mathbf{x}}_{t+N/t}, \hat{\mathbf{u}}_{t+N/t}) \leq \\ &\quad - \kappa \mathbf{l}(\hat{\mathbf{x}}_{t+N/t}, \hat{\mathbf{u}}_{t+N/t}) \end{aligned}$$

and so, from (3.29), it follows that

$$\mathbf{V}^{N,0}(\hat{\mathbf{x}}_{t+1/t}) \leq \mathbf{V}^{N,0}(\hat{\mathbf{x}}_{t/t}) - \mathbf{l}(\hat{\mathbf{x}}_{t/t}, \hat{\mathbf{u}}_{t/t}) - \kappa \mathbf{l}(\hat{\mathbf{x}}_{t+N/t}, \hat{\mathbf{u}}_{t+N/t}) \quad (3.30)$$

Recall the definition of l_i and of matrix R_i , for all $i = 1, \dots, M$, and define $\mathbf{R} = \text{diag}(R_1, \dots, R_M)$. Then, there exists a \mathcal{K}_∞ function α_L such that $\mathbf{l}(\hat{\mathbf{x}}, \hat{\mathbf{u}}) \geq \alpha_L(\|\hat{\mathbf{x}}, \mathbf{R}\hat{\mathbf{u}}\|)$ for all $\hat{\mathbf{x}} \in \mathbb{R}^n$, $\hat{\mathbf{u}} \in \mathbb{R}^m$. This implies that $\mathbf{l}(\hat{\mathbf{x}}, \hat{\mathbf{u}}) \geq \alpha_L(\|\hat{\mathbf{x}}\|)$ for all $\hat{\mathbf{x}} \in \mathbb{R}^n$, $\hat{\mathbf{u}} \in \mathbb{R}^m$. Therefore

$$\mathbf{V}^{N,0}(\hat{\mathbf{x}}_{t+1/t}) \leq \mathbf{V}^{N,0}(\hat{\mathbf{x}}_{t/t}) - \alpha_L(\|\hat{\mathbf{x}}_{t/t}\|) - \kappa \alpha_L(\|\tilde{\mathbf{x}}_{t+N}\|) \quad (3.31)$$

for all feasible sequences $\tilde{\mathbf{x}}_k$, $k = t, \dots, t+N-1$.

Now we analyze the properties of the cost function $\mathbf{V}^{N*}(\bar{\mathbf{x}}_t)$ defined in (3.24). First, note that, by definition of $\hat{\mathbf{x}}_{t/t}$, we have that $\mathbf{V}^{N*}(\bar{\mathbf{x}}_t) = \mathbf{V}^{N,0}(\hat{\mathbf{x}}_{t/t})$. By optimality, we have that

$$\mathbf{V}^{N*}(\bar{\mathbf{x}}_{t+1}) = \mathbf{V}^{N,0}(\hat{\mathbf{x}}_{t+1/t+1}) \leq \mathbf{V}^{N,0}(\hat{\mathbf{x}}_{t+1/t})$$

Considering (3.31), we obtain that

$$\mathbf{V}^{N*}(\bar{\mathbf{x}}_{t+1}) \leq \mathbf{V}^{N*}(\bar{\mathbf{x}}_t) - \alpha_L(\|\hat{\mathbf{x}}_{t/t}\|) - \kappa\alpha_L(\|\tilde{\mathbf{x}}_{t+N}\|) \quad (3.32)$$

for all $\bar{\mathbf{x}}_t \in \bar{\mathbb{X}}_{\mathbf{x}_t}^N$, being $\mathbf{x}_t \in \mathbb{X}^N$, and for all sequences $\tilde{\mathbf{x}}_{[t:t+N-1]} \in \tilde{\mathbb{X}}_{\mathbf{x}_t, \bar{\mathbf{x}}_t}^N$. This proves that $\|\hat{\mathbf{x}}_{t/t}\| \rightarrow 0$ and $\|\tilde{\mathbf{x}}_t\| \rightarrow 0$ as $t \rightarrow +\infty$.

3.6.4 Convergence of the trajectories

Let δ_F be a positive real number such that, if $\|\hat{\mathbf{x}}_k\| < \delta_F$, $\|\tilde{\mathbf{x}}_k\| < \delta_F$, $k = t, \dots, t+N$ and $\|\hat{\mathbf{u}}_k\| < \delta_F$, $k = t, \dots, t+N-1$, then constraints (3.25b)-(3.26) are satisfied.

Define a sequence $\mathbf{x}_{k/t}^*$, $k = t, \dots, t+N$, stemming from the initial condition $\mathbf{x}_{t/t}^* = \hat{\mathbf{x}}_{t/t}$, whose dynamics obeys to (3.21), and where the input is $\hat{\mathbf{u}}_k = \mathbf{u}_{k/t}^* = \mathbf{K}^{aux} \mathbf{x}_{k/t}^*$, for all $k = t, \dots, t+N-1$. Then there exists a positive real number $\delta_x < \delta_F$ such that, if $\|\hat{\mathbf{x}}_{t/t}\| < \delta_x$ and $\|\tilde{\mathbf{x}}_k\| < \delta_x$ for $k = t, \dots, t+N-1$, then $\|\mathbf{x}_{k/t}^*\| < \delta_F$, $k = t, \dots, t+N$, and $\|\mathbf{u}_{k/t}^*\| < \delta_F$, $k = t, \dots, t+N-1$. In fact, denoting $\mathbf{F} = \mathbf{A}^* + \mathbf{B}\mathbf{K}^{aux}$, from (3.21), for $i \geq 1$

$$\mathbf{x}_{t+i/t}^* = \mathbf{F}^i \hat{\mathbf{x}}_{t/t} + \sum_{j=0}^{i-1} \mathbf{F}^j \tilde{\mathbf{A}} \tilde{\mathbf{x}}_{t+i-j-1} \quad (3.33)$$

and $\|\mathbf{x}_{t+i/t}^*\| = \|\hat{\mathbf{x}}_{t/t}\| < \delta_x < \delta_F$, $\|\mathbf{x}_{t+i/t}^*\| < \max_{i=1, \dots, N} \|\mathbf{F}^i + \sum_{j=0}^{i-1} \mathbf{F}^j \tilde{\mathbf{A}}\| \delta_x$ and $\|\mathbf{u}_{k/t}^*\| \leq \|\mathbf{K}^{aux}\| \|\mathbf{x}_{k/t}^*\|$. Therefore, for a suitable δ_x , if $\|\hat{\mathbf{x}}_{t/t}\| < \delta_x$ and $\|\tilde{\mathbf{x}}_k\| < \delta_x$, $k = t, \dots, t+N-1$, then the trajectories $\mathbf{x}_{k/t}^*$, $k = t, \dots, t+N$ and $\mathbf{u}_{k/t}^*$, $k = t, \dots, t+N-1$ are feasible (since also $\hat{\mathbf{x}}_{t/t}$ satisfies (3.25a) for the feasibility of the i -DPC problem at time t).

Since $\|\hat{\mathbf{x}}_{t/t}\| \rightarrow 0$ and $\|\tilde{\mathbf{x}}_t\| \rightarrow 0$ as $t \rightarrow +\infty$, there exists $\bar{t} > 0$ such that $\|\hat{\mathbf{x}}_{t/t}\| < \delta_x$ and $\|\tilde{\mathbf{x}}_t\| < \delta_x$ for all $t \geq \bar{t}$, which makes the trajectories $\mathbf{x}_{k/t}^*$, $k = t, \dots, t+N$, and $\mathbf{u}_{k/t}^*$, $k = t, \dots, t+N-1$, feasible for all $t \geq \bar{t}$. By optimality, if $t \geq \bar{t}$

$$\mathbf{V}^{N*}(\bar{\mathbf{x}}_t) = \mathbf{V}^{N,0}(\hat{\mathbf{x}}_{t/t}) \leq \sum_{k=t}^{t+N-1} \mathbf{l}(\mathbf{x}_{k/t}^*, \mathbf{u}_{k/t}^*) + \mathbf{V}^F(\mathbf{x}_{t+N/t}^*) \quad (3.34)$$

Recall (3.18). Since $\mathbf{V}^F \geq 0$ by definition, one has that $\mathbf{l}(\mathbf{x}_{k/t}^*, \mathbf{u}_{k/t}^*) \leq \frac{1}{1+\kappa} \mathbf{V}^F(\mathbf{x}_{k/t}^*) \leq \mathbf{V}^F(\mathbf{x}_{k/t}^*)$ and, from (3.34)

$$\mathbf{V}^{N*}(\bar{\mathbf{x}}_t) \leq \sum_{k=t}^{t+N} \mathbf{V}^F(\mathbf{x}_{k/t}^*) \quad (3.35)$$

From (3.33) and (3.35), we obtain that, for all $t \geq \bar{t}$, there exists a \mathcal{H}_∞ function β such that

$$\mathbf{V}^{N*}(\bar{\mathbf{x}}_t) \leq \beta(\|(\hat{\mathbf{x}}_{t/t}, \tilde{\mathbf{x}}_{[t:t+N-1]})\|) \quad (3.36)$$

For this it follows that $\mathbf{V}^{N*}(\bar{\mathbf{x}}_t) \rightarrow 0$ as $t \rightarrow +\infty$.

Recall that $\hat{\mathbf{x}}_{k/t}$ is generated according to (3.21), stemming from the optimal initial condition $\hat{\mathbf{x}}_{t/t}$ and inputs $\hat{\mathbf{u}}_{k/t}$. One can write the solution to (3.21) as $\hat{\mathbf{x}}_{t+i/t} = \mathbf{v}_{t+i/t} + \mathcal{B}_i \mathbf{U}_t$, where

$$\mathbf{v}_{t+i/t} = (\mathbf{A}^*)^i \hat{\mathbf{x}}_{t/t} + \sum_{j=0}^{i-1} (\mathbf{A}^*)^j \tilde{\mathbf{A}} \hat{\mathbf{x}}_{t+i-j-1},$$

$$\mathcal{B}_i = [(\mathbf{A}^*)^{i-1} \mathbf{B} \quad \dots \quad \mathbf{B} \quad 0 \quad \dots \quad 0]$$

if $i = 1, \dots, N$, $\mathbf{U}_t = (\hat{\mathbf{u}}_{t/t}, \dots, \hat{\mathbf{u}}_{t+N-1/t})$. Note that, since $\|\hat{\mathbf{x}}_{t/t}\| \rightarrow 0$ and $\|\tilde{\mathbf{x}}_t\| \rightarrow 0$ as $t \rightarrow +\infty$, also $\|\mathbf{v}_{k/t}\| \rightarrow 0$ as $t \rightarrow +\infty$ for all $k = t+1, \dots, t+N$. We also denote $\mathbf{v}_{t/t} = \hat{\mathbf{x}}_{t/t}$ and $\mathcal{B}_0 = \mathbf{0}_{n \times Nm}$.

Now, consider again the function $\mathbf{V}^{N*}(\bar{\mathbf{x}}_t)$:

$$\mathbf{V}^{N*}(\bar{\mathbf{x}}_t) = \sum_{k=t}^{t+N-1} \mathbf{I}(\mathbf{v}_{k/t} + \mathcal{B}_{k-t} \mathbf{U}_t, \hat{\mathbf{u}}_{k/t}) + \mathbf{V}^F(\mathbf{v}_{t+N/t} + \mathcal{B}_N \mathbf{U}_t) \quad (3.37)$$

From the definition of l_i it follows that $\mathbf{I}(\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_k) \geq \alpha_L(\|(\hat{\mathbf{x}}_k, \mathbf{R}\hat{\mathbf{u}}_k)\|)$, and so

$$0 \leq \sum_{k=t}^{t+N-1} \alpha_L(\|(\mathbf{v}_{k/t} + \mathcal{B}_{k-t} \mathbf{U}_t, \mathbf{R}\hat{\mathbf{u}}_{k/t})\|) + \mathbf{V}^F(\mathbf{v}_{t+N/t} + \mathcal{B}_N \mathbf{U}_t) \leq \mathbf{V}^{N*}(\bar{\mathbf{x}}_t)$$

Since it is proved that $\mathbf{V}^{N*}(\bar{\mathbf{x}}_t) \rightarrow 0$ as $t \rightarrow +\infty$, it follows that, for all $k = t, \dots, t+N-1$

$$\alpha_L(\|(\mathbf{v}_{k/t} + \mathcal{B}_{k-t} \mathbf{U}_t, \mathbf{R}\hat{\mathbf{u}}_{k/t})\|) \rightarrow 0$$

and $\mathbf{V}^F(\mathbf{v}_{t+N/t} + \mathcal{B}_N \mathbf{U}_t) \rightarrow 0$ as $t \rightarrow +\infty$. This implies that:

$$\mathbb{B} \mathbf{U}_t + \mathbf{V}_t \rightarrow 0 \quad (3.38)$$

as $t \rightarrow \infty$, where

$$\mathbb{B} = \begin{bmatrix} \bar{\mathcal{B}} \\ \text{diag}(\mathbf{R}, \dots, \mathbf{R}) \end{bmatrix}, \quad \bar{\mathcal{B}} = [\mathcal{B}_0^T \quad \dots \quad \mathcal{B}_N^T]^T$$

and $\mathbf{V}_t = (v_{t/t}, \dots, v_{t+N/t}, 0, \dots, 0)$. It is readily seen that, in view of the triangular structure of $\bar{\mathcal{B}}$ and since, by definition of R_i , $i = 1, \dots, M$, $\text{rank} \begin{pmatrix} \mathbf{B}^T & \mathbf{R}^T \end{pmatrix}^T = m$ then $\text{rank}(\mathbb{B}) = Nm$. Since $\mathbf{V}_t \rightarrow 0$ as $t \rightarrow +\infty$, from (3.38) it follows that $\mathbf{U}_t \rightarrow 0$ as $t \rightarrow +\infty$. Therefore $\hat{\mathbf{u}}_{t/t} \rightarrow 0$ as $t \rightarrow +\infty$.

Finally, recall that the state \mathbf{x}_t and its estimate $\bar{\mathbf{x}}_t$ evolve according to the equations

$$\begin{cases} \mathbf{x}_{t+1} &= \mathbf{A} \mathbf{x}_t + \mathbf{B} [\hat{\mathbf{u}}_{t/t} + \mathbf{K}^{aux}(\bar{\mathbf{x}}_t - \hat{\mathbf{x}}_{t/t})] \\ \bar{\mathbf{x}}_{t+1} &= \mathbf{A} \bar{\mathbf{x}}_t + \mathbf{B} [\hat{\mathbf{u}}_{t/t} + \mathbf{K}^{aux}(\bar{\mathbf{x}}_t - \hat{\mathbf{x}}_{t/t})] - \mathbf{L} \mathbf{C}(\mathbf{x}_t - \bar{\mathbf{x}}_t) \end{cases}$$

Recalling that $\boldsymbol{\sigma}_t = \mathbf{x}_t - \bar{\mathbf{x}}_t$, the dynamics of $(\boldsymbol{\sigma}_t, \bar{\mathbf{x}}_t)$ is given by

$$\begin{cases} \boldsymbol{\sigma}_{t+1} &= (\mathbf{A} + \mathbf{L} \mathbf{C}) \boldsymbol{\sigma}_t \\ \bar{\mathbf{x}}_{t+1} &= (\mathbf{A} + \mathbf{B} \mathbf{K}^{aux}) \bar{\mathbf{x}}_t - \mathbf{L} \mathbf{C} \boldsymbol{\sigma}_t + \mathbf{B} (\hat{\mathbf{u}}_{t/t} - \mathbf{K}^{aux} \hat{\mathbf{x}}_{t/t}) \end{cases}$$

By asymptotic convergence to zero of the nominal state and input signals $\hat{\mathbf{x}}_{t/t}$ and $\hat{\mathbf{u}}_{t/t}$ respectively, we obtain that

$\mathbf{B}(\hat{\mathbf{u}}_{t/t} - \mathbf{K}^{aux} \hat{\mathbf{x}}_{t/t})$ is an asymptotically vanishing term. Since $(\mathbf{A} + \mathbf{B} \mathbf{K}^{aux})$ and $(\mathbf{A} + \mathbf{L} \mathbf{C})$ are Schur by Assumption 3 and 1, we obtain that $\boldsymbol{\sigma}_t \rightarrow 0$ and $\bar{\mathbf{x}}_t \rightarrow 0$ as $t \rightarrow +\infty$, from which it follows that $\mathbf{x}_t = \bar{\mathbf{x}}_t + \boldsymbol{\sigma}_t \rightarrow 0$ as $t \rightarrow +\infty$.

3.7 Conclusions

The output feedback distributed predictive control algorithm presented in this chapter has many features which make it suited for practical applications, such as the limited mutual knowledge and exchange of information among neighbors, the possibility to handle local and global state and control constraints, and guaranteed convergence properties.

However, a number of significant developments are required to completely exploit the potentialities of the approach in many significant practical cases. Among them, the solution of the tracking problem for constant reference signals and the possibility to include in the problem formulation joint (cooperative) goals for the subsystems will be considered in the near future.

Bibliography

- [1] I. Alvarado, D. Limon, D. Muñoz de la Peña, J.M. Maestre, M.A. Ridao, H. Sheu, W. Marquart, R.R. Negenborn, B. De Schutter, F. Valencia, and J. Espinosa. A comparative analysis of distributed mpc techniques applied to the hd-mpc four-tank benchmark. *Journal of Process Control*, 2011.
- [2] I. Alvarado, D. Limon, and Miguel A. Ridao. Documentation for benchmark cases-part I (HD-MPC D6.3.1 Part I). Public deliverable, HD-MPC Consortium, 2010. This deliverable and a detailed description of the plant and the simulation model can be downloaded from <http://www.ict-hd-mpc.eu>.
- [3] S. Dashkovskiy, B. S. Rüffer, and F.R. Wirth. An ISS small gain theorem for iss general networks. *Mathematics of Control, Signals, and Systems*, 19(2):93–122, 2007.
- [4] M. Farina and R. Scattolini. Distributed non-cooperative mpc with neighbor-to-neighbor communication. *Proceedings of the IFAC 2011 World Conference. August 28 - September 2 2011, Milan, Italy*, 2010. To appear.
- [5] M. Farina and R. Scattolini. Distributed predictive control: a non-cooperative algorithm with neighbor-to-neighbor communication. *Automatica*, 2010. Submitted.
- [6] K. H. Johansson. The quadruple-tank process. *IEEE Transactions on Control Systems Technology*, 8:456–465, 2000.
- [7] D.Q. Mayne, S.V. Rakovic, R. Findeisen, and F. Allgöwer. Robust output feedback model predictive control of constrained linear systems: Time varying case. *Automatica*, 43:2082–2087, 2009.
- [8] D.Q. Mayne, M.M. Seron, and S. V. Rakovic. Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, 41:219–224, 2005.
- [9] D. D. Šiljac. *Large-scale dynamic systems. Stability and structure*. North Holland, 1978.