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Executive Summary

This report is split into two parts and thus contains two different aspects of robust distributed MPC. Chapter 1, contains a "Robustness analysis of nominal Model Predictive Control for nonlinear discrete-time systems". Robust MPC methods are much more complex than those developed for nominal conditions, requiring either an heavy on-line computational burden, or a long off-line design phase. For this reason, it is still of interest to move back to the problem of analyzing under which conditions nominal MPC can guarantee robustness in the face of specific classes of disturbances. This is exactly the goal of the presented research in Chapter 1. There, Input-to-State Stability properties of a system subject to additive disturbances are studied under the assumption that the origin of the nominal system is an asymptotically stable equilibrium. The analysis is developed also for systems with a discontinuous dynamic equation. The obtained results are used to establish robustness properties for perturbed systems controlled with a Model Predictive Control law designed for the nominal model. It is shown that, under mild assumptions, the design of an MPC law for a nominal model guarantees also robustness in perturbed conditions. This is proved by first deriving a number of results for systems characterized by a not necessarily continuous dynamic equation and subject to additive disturbances. It is believed that these results can be of wider applicability, including the study of other control synthesis techniques than MPC. Thus, this can be seen as a basis for the application in hierarchical and distributed MPC.
Chapter 2 contains a method for "Fully Decentralized nominal MPC". In this work a fully decentralized MPC is considered, i.e. that there does not exist any information exchange. In this case, the possible interactions between subsystems are considered as unknown disturbances that the controller must accomplish. The design of a fully decentralized MPC can be done relying on a robust design of each predictive controller. The methodology to design the nominal MPC for each subsystem is presented. Under a certain design, the nominal MPC can ensure Input-to-State Stability (ISS) of the system. The uncertainty is modeled as a parametric uncertain signal, not as an additive disturbance. Assuming that the model function is uniformly continuous, enhanced design of the robust controller is achieved: in the calculation of the constraints of the optimization problem and in the stabilizing conditions. The obtained stabilizing design of the controller results particularly interesting to relax the terminal conditions for a certain class of model functions yielding to a less conservative control law. The controllers derived are appealing from a practical point of view since can be constructed from standard nominal MPC. On the other hand, the open-loop nature of the problem may yield to the results to be useful only for small uncertainties. In order to reduce this effect, semi-feedback approach is proposed.

Chapter 1

Robustness analysis of nominal Model Predictive Control for nonlinear discrete-time systems

The contents of this chapter have been developed by Bruno Picasso, Delia Desiderio and Riccardo Scattolini (Politecnico di Milano, Dipartimento di Elettronica e Informazione). The results will be presented at "Nolcos 2010", September, Bologna.

Abstract

Input-to-State Stability properties of a system subject to additive disturbances are studied under the assumption that the origin of the nominal system is an asymptotically stable equilibrium. The analysis is developed also for systems with a discontinuous dynamic equation. The obtained results are used to establish robustness properties for perturbed systems controlled with a Model Predictive Control law designed for the nominal model.

1.1 Introduction

Model Predictive Control (MPC) is a technique widely applied in the process industry in view of its capability to explicitly consider state and control constraints, as well as to deal with very large scale problems with hundreds of control and controlled variables, see [16] and references therein. There are nowadays many ways to formulate stabilizing MPC methods, see e.g. [13] for a survey on this topic. However, it is also well known that nominal MPC can be non robust with respect to even arbitrarily small disturbances, see [2]. Moreover, as it is discussed in [5, 8], discontinuity of the closed-loop dynamics, and of the Lyapunov functions for the nominal system, can emphasize such a lack of robustness. This issue is crucial in MPC, where both the resulting feedback law and the available Lyapunov function (which is typically the value function associated to the optimal control problem defining MPC) can be discontinuous (see also [15]). For this reason, in the last years, attention has been focused on the development of MPC algorithms robust with respect to specific classes of disturbances, see the review papers [12, 17]. This activity has lead to the development of two broad classes of robust MPC algorithms. The first one is based on a min-max formulation of the underlying optimization problem (see, inter alia, the synthesis method described in [10]); the second class

of algorithms is based on the a-priori evaluation of the effect of the disturbance over the prediction horizon and on the use of tighter and tighter constraints to be imposed in the optimization problem to the predicted state trajectories. Examples of this approach are reported in [6, 14] and, for nominal MPC, in [3].
In any case, robust MPC methods are much more complex than those developed for nominal conditions, requiring either an heavy on-line computational burden, or a long off-line design phase. For this reason, it is still of interest to move back to the problem of analyzing under which conditions nominal MPC can guarantee robustness in the face of specific classes of disturbances. This is exactly the goal of this note, which is organized as follows. In Section 1.2, the system is defined and the concepts of Input-to-State Stability (ISS) and Input-to-State practical Stability (ISpS) are recalled together with some related recent results, see [7, 4, 9, 11]. Section 1.3 presents the main results of the paper concerning the characterization of stability properties in perturbed conditions which can be deduced by the properties of a Lyapunov function for the nominal system. More specifically, a function Ψ is constructed in terms of standard $\mathscr{K}_\infty$-functions used to bound V and its variation along trajectories [7]. The robustness analysis is then easily derived by the analysis of the behavior of such a function Ψ, which thus represents a kind of robustness energy measure. The analysis includes the critical case of systems with a discontinuous dynamic equation and discontinuous Lyapunov functions. A classical example, already considered in [5, 8], is revisited in view of the results reported in this paper. Section 1.4 specializes the analysis to linear systems. Finally, in Section 1.5, the achieved results are applied to closed-loop dynamics resulting by an MPC stabilizing the nominal system. In particular, it is proved that, under mild and easily testable assumptions, robustness properties can be enforced by properly selecting the free tuning parameters of an MPC algorithm designed for the nominal model. Some conclusions close the paper.
Notation and terminology: The sets of non negative integers and real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}_+$, respectively. Let $GL(n,\mathbb{R}) = \{T \in \mathbb{R}^{n\times n} \,|\, \det T \neq 0\}$.
A generic vector norm in $\mathbb{R}^h$ is denoted by $|\cdot|$. If $\mathbb{R}^{n\times n} \ni P > 0$, we let $|x|_P = \sqrt{x'Px}$ and, for $A \in \mathbb{R}^{n\times n}$, $|A|_P = \max_{x:|x|_P=1} |Ax|_P$ be the corresponding induced matrix norm. The Euclidean vector or induced matrix norm is denoted by $|\cdot|_2$. A signal taking values in $\mathbb{R}^n$ is denoted by $\mathbf{w} = \{w(0), w(1), \ldots\}$. For $\mathscr{W} \subseteq \mathbb{R}^n$, let $\mathscr{M}_\mathscr{W}$ be the set of signals taking values in $\mathscr{W}$ and such that $\|\mathbf{w}\| = \sup_{k\in\mathbb{N}} |w(k)| < +\infty$. The interior part of a set $\mathscr{S} \subseteq \mathbb{R}^n$ is denoted by $int(\mathscr{S})$. For any $r > 0$, let $\mathscr{B}_r = \{x \in \mathbb{R}^n : |x| \leq r\}$.
A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathscr{K}$-function iff $\alpha(0) = 0$ and it is strictly increasing. A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathscr{K}_\infty$-function iff it is a $\mathscr{K}$-function and $\alpha(s) \to +\infty$ as $s \to +\infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{N} \to \mathbb{R}_+$ is a $\mathscr{KL}$-function iff, for any fixed $k \geq 0$, $\beta(\cdot,k)$ is a $\mathscr{K}$-function in s and, for each fixed $s > 0$, $\beta(s,\cdot)$ is decreasing and $\beta(s,k) \to 0$ as $k \to +\infty$. The identity function $s \mapsto s$ is denoted by Id.

1.2 Preliminaries: system definition and robust stability properties

Let the nominal nonlinear discrete-time model be

$$x(k+1) = f\big(x(k)\big), \quad k \in \mathbb{N}, \quad x(0) = \bar{x}, \tag{1.1}$$

where $x(k) \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$, with $f(0) = 0$, not necessarily a continuous function. Suppose that the perturbed system takes the form

$$\begin{aligned} x(k+1) = &\ \tilde{f}\big(x(k), w(k)\big) = \\ = &\ f\big(x(k)\big) + w(k), \quad k \in \mathbb{N}, \quad x(0) = \bar{x}, \end{aligned} \tag{1.2}$$

with $w(k) \in \mathscr{W} \subseteq \mathbb{R}^n$. The solution of system (1.2) at time k for $x(0) = \bar{x}$ and disturbance $\mathbf{w}$ is denoted by $x(k, \bar{x}, \mathbf{w})$.
Consider the nominal system (1.1). The classical Lyapunov theorem states that if a continuous positive function V exists such that $\Delta V(x) = V(f(x)) - V(x)$ is continuous and negative definite, then the origin is an asymptotically stable equilibrium. However, if the function f is discontinuous, the continuity of both V and ΔV is not guaranteed in general and $\Delta V < 0$ is not sufficient for the asymptotic stability. Therefore, both the Lyapunov function definition and the stability theorem have to be suitably modified.

Definition 1 *A set $\Gamma \subseteq \mathbb{R}^n$ is said to be positively invariant for system* (1.1) *iff $\forall x \in \Gamma$, $f(x) \in \Gamma$.*

Definition 2 *A function $V : \mathbb{R}^n \to \mathbb{R}_+$ is a Lyapunov function for system* (1.1) *iff there exist two sets Ω and Γ with $\Omega \subseteq \Gamma$, $0 \in int(\Omega)$, and $\mathscr{K}_\infty$-functions α_1, α_2 and α_3 such that:*

$$V(x) \geq \alpha_1(|x|) \quad \forall x \in \Gamma \tag{1.3a}$$
$$V(x) \leq \alpha_2(|x|) \quad \forall x \in \Omega \tag{1.3b}$$
$$V(f(x)) - V(x) \leq -\alpha_3(|x|) \quad \forall x \in \Gamma. \tag{1.3c}$$

Proposition 1 *[7] Let V be a Lyapunov function for system* (1.1) *and assume that Γ is positively invariant. Then the origin is an asymptotically stable equilibrium in Γ.*

For perturbed systems (1.2), stability notions capable of taking the effect of disturbances $\mathbf{w}$ into account occur.

Definition 3 *A set $\Gamma \subseteq \mathbb{R}^n$ is said to be robust positively invariant with respect to $\mathscr{W} \subseteq \mathbb{R}^n$ ($\mathscr{W}$-RPI) for system* (1.2) *iff $\forall x \in \Gamma$ and $\forall w \in \mathscr{W}$, $\tilde{f}(x,w) \in \Gamma$.*

Definition 4 *Given a compact set $\Gamma \subseteq \mathbb{R}^n$ with $0 \in int(\Gamma)$, system* (1.2) *is said to be Input-to-State practically Stable in Γ with respect to $\mathscr{W} \subseteq \mathbb{R}^n$ $\big((\Gamma,\mathscr{W})$-ISpS$\big)$ iff Γ is a $\mathscr{W}$-RPI set for* (1.2) *and there exist a $\mathscr{K}\mathscr{L}$-function β, a $\mathscr{K}$-function γ and a constant $c \geq 0$ such that $\forall k \geq 0$, $\forall \bar{x} \in \Gamma$ and $\forall \mathbf{w} \in \mathscr{M}_{\mathscr{W}}$,*

$$|x(k,\bar{x},\mathbf{w})| \leq \beta(\bar{x},k) + \gamma(\|\mathbf{w}\|) + c. \tag{1.4}$$

According to inequality (1.4), in the case of vanishing disturbance signals, only convergence of the state trajectory towards the neighborhood $\mathscr{B}_c$ of the origin is guaranteed.

Definition 5 *A function $V : \mathbb{R}^n \to \mathbb{R}_+$ is a $(\Gamma,\mathscr{W})$-ISpS Lyapunov function for system* (1.2) *iff Γ is a compact $\mathscr{W}$-RPI set and there exist a compact set $\Omega \subseteq \Gamma$ with $0 \in int(\Omega)$, some $\mathscr{K}_\infty$-functions $\alpha_1, \alpha_2, \alpha_3$, a $\mathscr{K}$-function σ and constants $c_1, c_2 \geq 0$ such that:*

$$V(x) \geq \alpha_1(|x|) \quad \forall x \in \Gamma \tag{1.5a}$$
$$V(x) \leq \alpha_2(|x|) + c_1 \quad \forall x \in \Omega \tag{1.5b}$$
$$V(\tilde{f}(x,w)) - V(x) \leq -\alpha_3(|x|) + \sigma(|w|) + c_2 \quad \forall x \in \Gamma \; \forall w \in \mathscr{W}. \tag{1.5c}$$

Proposition 2 *[9] If* (1.2) *admits a $(\Gamma,\mathscr{W})$-ISpS Lyapunov function, then it is $(\Gamma,\mathscr{W})$-ISpS.*

Definition 6 *Given a compact set $\Gamma \subset \mathbb{R}^n$ with $0 \in int(\Gamma)$, system* (1.2) *is said to be Input-to-State Stable in Γ with respect to $\mathscr{W} \subseteq \mathbb{R}^n$ $\big((\Gamma,\mathscr{W})$-ISS$\big)$ iff Γ is $\mathscr{W}$-RPI for system* (1.2) *and there exist a $\mathscr{K}\mathscr{L}$-function β and a $\mathscr{K}$-function γ such that $\forall k \geq 0$, $\forall \bar{x} \in \Gamma$ and $\forall \boldsymbol{w} \in \mathscr{M}_{\mathscr{W}}$,*

$$|x(k,\bar{x},\boldsymbol{w})| \leq \beta(\bar{x},k) + \gamma(\|\boldsymbol{w}\|).$$

Definition 7 *A function $V : \mathbb{R}^n \to \mathbb{R}_+$ is a $(\Gamma,\mathscr{W})$-ISS Lyapunov function for system* (1.2) *iff Γ is a compact $\mathscr{W}$-RPI set and there exist a compact set $\Omega \subseteq \Gamma$ with $0 \in int(\Omega)$, some $\mathscr{K}_\infty$-functions $\alpha_1, \alpha_2, \alpha_3$ and a $\mathscr{K}$-function σ such that:*

$$V(x) \geq \alpha_1(|x|) \quad \forall x \in \Gamma \tag{1.6a}$$

$$V(x) \leq \alpha_2(|x|) \quad \forall x \in \Omega \tag{1.6b}$$

$$V(\tilde{f}(x,w)) - V(x) \leq -\alpha_3(|x|) + \sigma(|w|) \quad \forall x \in \Gamma\, \forall w \in \mathscr{W}. \tag{1.6c}$$

Proposition 3 *[11] If* (1.2) *admits a $(\Gamma,\mathscr{W})$-ISS Lyapunov function, then it is $(\Gamma,\mathscr{W})$-ISS.*

1.3 Stability analysis in perturbed conditions

The stability properties for the perturbed system (1.2) are now analyzed assuming that the origin is an asymptotically stable equilibrium for the nominal model (1.1). The proposed results need only the knowledge of the $\mathscr{K}_\infty$-functions α_i's ($i = 1,2,3$) in inequalities (1.3) and associated with a Lyapunov function for the nominal system.

Definition 8 *Let V be a Lyapunov function for the nominal system* (1.1). *For $s \geq 0$, let $\psi(s) = \max_{\varsigma \in [0,s]}(\alpha_2 - \alpha_3)(\varsigma)$, then*

$$\Psi(s) = (Id - \alpha_1^{-1} \circ \psi)(s)$$

is called the Ψ-function associated to V.

Remark 1 *Notice that:*

1. *$(\alpha_1^{-1} \circ \psi)(s)$, and hence $\Psi(s)$, is well defined because*

$$\psi(s) \geq \psi(0) = (\alpha_2 - \alpha_3)(0) = 0;$$

2. *if $\alpha_2 - \alpha_3$ is a non decreasing function, $\psi = \alpha_2 - \alpha_3$;*

3. *Ψ is a continuous function;*

4. *indeed, the Ψ-function is depending on $(\alpha_1, \alpha_2, \alpha_3)$ and it is not univocally determined by V but, for the sake of brevity, it is referred to as "associated to V".*

Theorem 1 (RPI analysis) *Let V be a Lyapunov function for the nominal system* (1.1) *and assume that Γ is positively invariant. Let $r > 0$ be such that $\mathscr{B}_r \subseteq \Omega$. If $\Psi(r) > 0$, where Ψ is the Ψ-function associated to V, then $\mathscr{B}_r$ is a $\mathscr{B}_\mu$-RPI set for the perturbed system* (1.2) *with*

$$\mu = \Psi(r). \tag{1.7}$$

Proof. We have to show that, $\forall x \in \mathscr{B}_r$ and $\forall w \in \mathscr{B}_\mu$, $\tilde{f}(x,w) \in \mathscr{B}_r$. For $x \in \mathscr{B}_r$ and $w \in \mathscr{B}_\mu$, it holds that

$$|\tilde{f}(x,w)| \leq |f(x)| + |w| \leq |f(x)| + \mu. \tag{1.8}$$

For $x \in \mathscr{B}_r$, by inequalities (1.3c) and (1.3b), one has

$$V(f(x)) \leq V(x) - \alpha_3(|x|) \leq \alpha_2(|x|) - \alpha_3(|x|), \tag{1.9}$$

where the last inequality can be applied because $x \in \mathscr{B}_r \subseteq \Omega$. By the positive invariance of Γ, $f(x) \in \Gamma$ so that, by (1.3a) and (1.9), $\alpha_1(|f(x)|) \leq V(f(x)) \leq (\alpha_2 - \alpha_3)(|x|)$ whence

$$|f(x)| \leq \big(\alpha_1^{-1} \circ (\alpha_2 - \alpha_3)\big)(|x|). \tag{1.10}$$

Thus, combining inequalities (1.8) and (1.10),

$$|\tilde{f}(x,w)| \leq \big(\alpha_1^{-1} \circ (\alpha_2 - \alpha_3)\big)(|x|) + \mu.$$

Therefore, $\tilde{f}(x,w) \in \mathscr{B}_r$ if

$$\forall x \in \mathscr{B}_r, \quad \big(\alpha_1^{-1} \circ (\alpha_2 - \alpha_3)\big)(|x|) + \mu \leq r,$$

which in turn is equivalent to

$$\begin{aligned} \mu \leq \quad & r - \max_{x \in \mathscr{B}_r} \big(\alpha_1^{-1} \circ (\alpha_2 - \alpha_3)\big)(|x|) = \\ = \quad & r - \alpha_1^{-1}\big(\max_{\varsigma \in [0,r]} (\alpha_2 - \alpha_3)(\varsigma)\big) = \\ = \quad & r - (\alpha_1^{-1} \circ \psi)(r) = \Psi(r) \end{aligned}$$

and this holds in view of equation (1.7). ■

Remark 2 *Notice that, for $s > 0$ such that $\mathscr{B}_s \subseteq \Omega$, $(\alpha_2 - \alpha_3)(s) \geq 0$, in fact for $x \in \Omega$ by* (1.9) *one has,*

$$(\alpha_2 - \alpha_3)(|x|) \geq V(f(x)) \geq 0.$$

An ISpS property can be obtained under the following mild assumption on the $\mathscr{K}_\infty$-function α_1 associated with the Lyapunov function V for the nominal model.

Assumption 1 *Let α_1 be so that $\exists \sigma \in \mathscr{K}_\infty$ such that*

$$\big|\alpha_1(|x|) - \alpha_1(|y|)\big| \leq \sigma(|x-y|) \quad \forall x,y \in \Omega. \tag{1.11}$$

This holds, for instance, if α_1 is Lipschitz continuous in Ω.
The ISpS result is based on the following properties:

Lemma 1 *Let V be a Lyapunov function for system* (1.1) *with Ω being a bounded set. Under Assumption 1 one has:*

a. *the function V is such that*

$$\big|V(x_1) - V(x_2)\big| \leq \sigma(|x_1 - x_2|) + d \quad \forall x_1, x_2 \in \Omega, \tag{1.12}$$

where

$$d = \sup_{x \in \Omega} \big(\alpha_2(|x|) - \alpha_1(|x|)\big); \tag{1.13}$$

b. *if $x \in \Omega$ and $w \in \mathscr{W}$ are such that $f(x) \in \Omega$ and $\tilde{f}(x,w) \in \Omega$, then*

$$V(\tilde{f}(x,w)) - V(x) \le -\alpha_3(|x|) + \sigma(|w|) + d.$$

Proof. The proof is easy and it is omitted. ■

Theorem 2 (ISpS analysis) *Let V be a Lyapunov function for the nominal system* (1.1), *assume that Γ is positively invariant and Ω is a bounded set. Consider $r > 0$ such that $\mathscr{B}_r \subseteq \Omega$ and $\Psi(r) > 0$, where Ψ is the Ψ-function associated to V. Then, under Assumption 1, the perturbed system* (1.2) *is $(\mathscr{B}_r, \mathscr{B}_\mu)$-ISpS with $\mu = \Psi(r)$. Moreover, V is a $(\mathscr{B}_r, \mathscr{B}_\mu)$-ISpS Lyapunov function.*

Proof. First notice that, by Theorem 1, $\mathscr{B}_r$ is a (compact) $\mathscr{B}_\mu$-RPI set. Let us show that V satisfies inequalities (1.5) with $\Omega = \Gamma = \mathscr{B}_r$, α_1, α_2 and α_3 as in Definition 2, σ as in Assumption 1 and suitable choices for c_j's ($j = 1,2$). The thesis then follows by Proposition 2.
Properties (1.5a) and (1.5b), that is

$$\begin{aligned} V(x) &\ge \alpha_1(|x|) \quad \forall x \in \mathscr{B}_r \\ V(x) &\le \alpha_2(|x|) + c_1 \quad \forall x \in \mathscr{B}_r, \end{aligned}$$

follow by inequalities (1.3a) and (1.3b), respectively, because $\mathscr{B}_r \subseteq \Omega \subseteq \Gamma$. Thus, property (1.5b) holds with $c_1 = 0$.
Property (1.5c), that is

$$\begin{aligned} V(\tilde{f}(x,w)) - V(x) \le -\alpha_3(|x|) + \sigma(|w|) + c_2 \\ \forall x \in \mathscr{B}_r \; \forall w \in \mathscr{B}_\mu, \end{aligned}$$

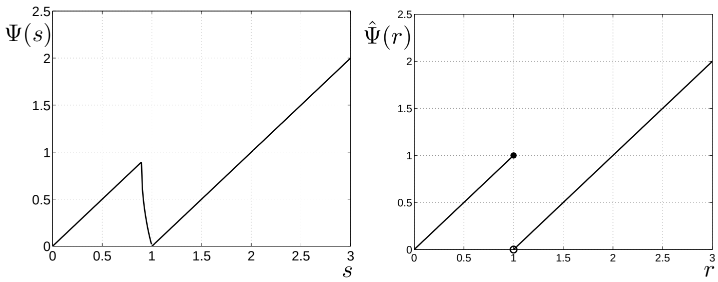
follows by Lemma 1.**b** and hence holds with σ and $c_2 = d$ given in equations (1.11) and (1.13), respectively. It is sufficient to notice that such a lemma can be applied because $\mathscr{B}_r$ is a $\mathscr{B}_\mu$-RPI set contained in Ω. ■
Finally, the most interesting property of ISS is analyzed.

Theorem 3 (ISS analysis) *Let V be a Lyapunov function for the nominal system* (1.1) *and assume that Γ is positively invariant. Let $\bar{r}$ be such that $\mathscr{B}_{\bar{r}} \subseteq \Omega$. If there exists a $\mathscr{K}_\infty$-function $\tilde{\Psi}$ such that $\forall r \le \bar{r}$, $\Psi(r) \ge \tilde{\Psi}(r)$, where Ψ is the Ψ-function associated to V, then, $\forall r \in (0, \bar{r}]$ the perturbed system* (1.2) *is $(\mathscr{B}_r, \mathscr{B}_\mu)$-ISS with $\mu = \Psi(r)$ and $\tilde{V}(x) = |x|$ is a $(\mathscr{B}_r, \mathscr{B}_\mu)$-ISS Lyapunov function.*

Proof. Since $\tilde{\Psi}$ is a $\mathscr{K}_\infty$-function, then $\Psi(r) \ge \tilde{\Psi}(r) > 0$. Hence, by Theorem 1, $\mathscr{B}_r$ is a $\mathscr{B}_\mu$-RPI set. Let us show that $\tilde{V}(x) = |x|$ satisfies inequalities (1.6) with $\Omega = \Gamma = \mathscr{B}_r$ and $\alpha_1 = \alpha_2 = Id$, $\alpha_3 = \tilde{\Psi}$ and $\sigma = Id$. The thesis then follows by Proposition 3.
Inequalities (1.6a) and (1.6b) are trivial, let us then consider inequality (1.6c). For $x \in \mathscr{B}_r$ and $w \in \mathscr{B}_\mu$, one has

$$\begin{aligned} \tilde{V}(\tilde{f}(x,w)) - \tilde{V}(x) &\le |f(x)| - |x| + |w| \le \\ &\overset{(a)}{\le} \left(\alpha_1^{-1} \circ (\alpha_2 - \alpha_3)\right)(|x|) - |x| + |w| \le \\ &\overset{(b)}{\le} -\left(Id - (\alpha_1^{-1} \circ \psi)\right)(|x|) + |w| = \\ &= -\Psi(|x|) + |w| \le -\tilde{\Psi}(|x|) + |w|, \end{aligned}$$

where (a) follows by inequality (1.10), which can be applied because $\mathscr{B}_r$ is a $\mathscr{B}_\mu$-RPI set contained in Ω, and (b) holds because $(\alpha_2 - \alpha_3)(s) \le \psi(s)$ and α_1^{-1} is a $\mathscr{K}_\infty$-function. ■
Notice that $\tilde{V}$ is a continuous function while V is not necessarily continuous.
Finally, for state-dependent disturbances, asymptotic state convergence to the origin can be obtained as it is stated in the following small-gain type result.



(a) (b)

Figure 1.1: (a) The Ψ-function associated to V; (b) Graph of the function $\hat{\Psi}$.

Corollary 1 *Under the hypotheses of Theorem 3, if $x(0) \in \mathscr{B}_r$ and $\exists\, \theta \in [0,1)$ such that, $\forall k \in \mathbb{N}$, $|w(k)| \leq \theta \Psi(|x(k)|)$, then the state trajectory converges to the origin. In particular, if $w = w(x)$, then the origin is an asymptotically stable equilibrium in $\mathscr{B}_r$.*

Proof. It easily follows by Theorem 3. ■

Example 1 *Consider system* (1.2) *where $x \in \mathbb{R}$ and*

$$f(x) = \begin{cases} 1 & x > 1 \\ 0 & x \leq 1. \end{cases}$$

The nominal system $x(k+1) = f\big(x(k)\big)$ admits the discontinuous Lyapunov function given by

$$V(x) = \begin{cases} x^2 + 1 & x > 1 \\ x^2 & x \leq 1 \end{cases}$$

and hence nominal asymptotic stability of the origin holds. Indeed, it is straightforward to see that the following $\mathscr{K}_\infty$-functions are such that inequalities (1.3) *are satisfied*[1] *(with $|x|$ being the absolute value of x):*

$$\alpha_1(s) = \alpha_3(s) = s^2$$
$$\alpha_2(s) = \begin{cases} s^2 & s \leq \sqrt{1-\varepsilon} \\ \left(1 + \frac{1}{\varepsilon}\right) s^2 + 1 - \frac{1}{\varepsilon} & \sqrt{1-\varepsilon} \leq s \leq 1 \\ s^2 + 1 & s \geq 1, \end{cases}$$

where $\varepsilon > 0$ is any fixed and arbitrarily small constant.
This example has been studied in [5] and [8], where critical issues on the robustness properties have been pointed out. Let us see which properties can be deduced by the proposed analysis based on the Ψ-function associated to V. Elementary computations allow one to determine the Ψ-function that turns out to be

$$\Psi(s) = \begin{cases} s & s \leq \sqrt{1-\varepsilon} \\ s - \sqrt{\frac{1}{\varepsilon} s^2 + 1 - \frac{1}{\varepsilon}} & \sqrt{1-\varepsilon} \leq s \leq 1 \\ s - 1 & s \geq 1, \end{cases}$$

see Figure 1.1.a. Hence, according to Theorem 1, $\forall r > 0$, $r \neq 1$, $\mathscr{B}_r$ is $\mathscr{B}_\mu$-RPI with $\mu = \Psi(r)$. Since $\Psi(1) = 0$, for $r = 1$ no RPI properties for $\mathscr{B}_r$ can be proved by Theorem 1. Furthermore, it

[1] To this end, notice that $\forall x \in \mathbb{R}$, $V\big(f(x)\big) - V(x) = -x^2$.

is easy to show that $\Psi(1) \leq 0$ *for any choice of the* $\mathscr{K}_\infty$*-functions* α_i*'s satisfying inequalities* (1.3). *Nevertheless, in this example, it is possible to exactly characterize the RPI properties of any closed ball* $\mathscr{B}_r$ *and to find that,* $\forall r > 0$, $\mathscr{B}_r$ *is* $\mathscr{B}_\mu$*-RPI with* $\mu = \hat{\Psi}(r)$*, where*

$$\hat{\Psi}(r) = \begin{cases} r & r \leq 1 \\ r-1 & r > 1 \end{cases}$$

(see Figure 1.1.b). Since $\hat{\Psi}(r)$ *is the maximal disturbance amplitude so that the robust positive invariance of* $\mathscr{B}_r$ *is preserved, then* $\Psi(r) \leq \hat{\Psi}(r)$*. Moreover,* Ψ *is a continuous function (see Remark 1). Thus, one can think of* Ψ *as a continuous lower approximation of the discontinuous function* $\hat{\Psi}$*. This fact clarifies the reason why RPI properties cannot be deduced by Theorem 1 for the closed ball* $\mathscr{B}_1$*.*
As for the ISS analysis, since $\Psi(1) = 0$*, according to Theorem 3 a lower bound* $\tilde{\Psi} \in \mathscr{K}_\infty$ *for the* Ψ *function does not exist on intervals* $[0, \bar{r}]$ *with* $\bar{r} \geq 1$*. On the other hand, the same result allows one to see that the perturbed system is* $(\mathscr{B}_r, \mathscr{B}_{\Psi(r)})$*-ISS* $\forall r < 1$*.*
For $r > 1$*, instead, only ISpS properties hold. Theorem 2, with* $\Gamma = \Omega = \mathscr{B}_r$*, shows that the system is* $(\mathscr{B}_r, \mathscr{B}_{\Psi(r)})$*-ISpS. The same result also returns the values* $c_1 = 0$ *and* $c_2 = d = 1$ *for the constants appearing in inequalities* (1.5b) *and* (1.5c) *whence it follows that, for disturbance signals* $\mathbf{w}$ *such that* $\lim_{k\to+\infty} w(k) = 0$*, trajectories converge to the closed ball* $\mathscr{B}_1$ *in the sense that* $\limsup_{k\to+\infty} |x(k)| \leq 1$*.*
♣

1.4 Linear systems

The results derived in the previous section are now specialized to perturbed stable linear systems described by

$$x(k+1) = \tilde{f}\big(x(k), w(k)\big) = Ax(k) + w(k), \tag{1.14}$$

where $A \in \mathbb{R}^{n\times n}$ is a Schur matrix. It is well known that asymptotically stable linear systems enjoy ISS properties. Let us show that such properties do follow by Theorem 3.
A Lyapunov function for the nominal system $x(k+1) = Ax(k)$ is $V(x) = x'Px$, where $P > 0$ is such that

$$A'PA - P = -Q < 0.$$

Let us derive feasible choices for the $\mathscr{K}_\infty$ functions α_i's, $i = 1,2,3$, so that inequalities (1.3) hold. To this end, let us consider the vector norm $|\cdot|_P$. With this choice, $\alpha_1(s) = \alpha_2(s) = s^2$ are so that inequalities (1.3a) and (1.3b) are satisfied $\forall x \in \mathbb{R}^n$. As for α_3, one can consider $\alpha_3(s) = cs^2$ for some suitable $c > 0$. In fact, since

$$\begin{aligned} V(Ax) - V(x) \leq -c|x|_P^2 \quad \forall x \in \mathbb{R}^n \;&\Leftrightarrow \\ \Leftrightarrow -x'Qx \leq -cx'Px \quad \forall x \in \mathbb{R}^n \;&\Leftrightarrow \\ \Leftrightarrow\; c \leq \tfrac{x'Qx}{x'Px} \quad \forall x \in \mathbb{R}^n \setminus \{0\}& \end{aligned}$$

and $\inf_{x\in\mathbb{R}^n\setminus\{0\}} \frac{x'Qx}{x'Px} \geq \frac{\lambda_{\min}(Q)}{\lambda_{\mathrm{Max}}(P)}$, then $c = \frac{\lambda_{\min}(Q)}{\lambda_{\mathrm{Max}}(P)}$ is a feasible choice for c. Other feasible choices are

$$c = \frac{\lambda_{\min}\big((T^{-1})'QT^{-1}\big)}{\lambda_{\mathrm{Max}}\big((T^{-1})'PT^{-1}\big)}, \tag{1.15}$$

where $T \in GL(n,\mathbb{R})$ is a free parameter. In fact, letting $\hat{x} = Tx$, one has

$$\inf_{x\in\mathbb{R}^n\setminus\{0\}} \tfrac{x'Qx}{x'Px} = \inf_{\hat{x}\in\mathbb{R}^n\setminus\{0\}} \tfrac{\hat{x}'(T^{-1})'QT^{-1}\hat{x}}{\hat{x}'(T^{-1})'PT^{-1}\hat{x}}.$$

Moreover, it is not hard to see that the maximal feasible value of c is obtained by taking T such that $T'T = P$ and, in this case, one has

$$\frac{\lambda_{\min}\big((T^{-1})'QT^{-1}\big)}{\lambda_{\mathrm{Max}}\big((T^{-1})'PT^{-1}\big)} = \inf_{x\in\mathbb{R}^n\setminus\{0\}} \frac{x'Qx}{x'Px} = 1-|A|_P^2. \tag{1.16}$$

Let us compute the corresponding Ψ-function associated to V. Since $(\alpha_2-\alpha_3)(s) = (1-c)s^2$ and, according to Remark 2, $(\alpha_2-\alpha_3)(s) \geq 0\ \forall s \geq 0$, then $c \leq 1$ and $\psi(s) = (1-c)s^2$. Thus,

$$\Psi(s) = (Id - \alpha_1^{-1} \circ \psi)(s) = (1-\sqrt{1-c})s$$

and $\Psi \in \mathcal{K}_\infty$ because, by $c > 0$, one has $(1-\sqrt{1-c}) > 0$.
Hence, by Theorem 3, the perturbed system (1.14) is $(\mathcal{B}_r, \mathcal{B}_\mu)$-ISS $\forall r > 0$, with

$$\mu = \Psi(r) = (1-\sqrt{1-c})r. \tag{1.17}$$

Remark 3 *Notice that, according to equation* (1.17)*, the smaller* $\sqrt{1-c}$ *is, the more robustness is ensured for the perturbed system* (1.14)*. By equation* (1.16) *and standard properties of the induced matrix norms, it holds that*

$$\rho(A) \leq |A|_P \leq \sqrt{1-c}.$$

This means that, according to the proposed theory, the spectral radius of A provides a restriction on the degree of robustness which can be proved.

Remark 4 *The choice of a proper vector norm is crucial for the applicability of the proposed results. For instance, if we consider* $V(x) = x'Px$ *and the Euclidean vector norm* $|\cdot|_2$*, then inequalities* (1.3) *are satisfied with* $\alpha_1(s) = \lambda_{\min}(P)s^2$*,* $\alpha_2(s) = \lambda_{\mathrm{Max}}(P)s^2$ *and* $\alpha_3(s) = \lambda_{\min}(Q)s^2$*. Hence, it is easy to see that*

$$\Psi(s) = \left(1-\sqrt{\tfrac{\lambda_{\mathrm{Max}}(P)-\lambda_{\min}(Q)}{\lambda_{\min}(P)}}\right)s.$$

In this case, however, it is not guaranteed that $\Psi \in \mathcal{K}_\infty$*. If* $\Psi \in \mathcal{K}_\infty$*, then robust positive invariance properties of the Euclidean balls hold. Therefore, a necessary condition in order that* $\Psi \in \mathcal{K}_\infty$ *is* $A'A - I < 0$ *(namely, that the system is 2-norm contractive).*

1.5 Application to Model Predictive Control

The proposed analysis results are now applied to systems controlled with an MPC law designed on the nominal model of a perturbed system. The goal is to provide MPC synthesis methods based on the nominal model and capable of guaranteeing robustness properties for the corresponding closed-loop system against additive disturbances.
Let the perturbed open-loop system be given by

$$x(k+1) = f_o\big(x(k), u(k)\big) + w(k),\ k \in \mathbb{N},\ x(0) = \bar{x}, \tag{1.18}$$

with $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $w(k) \in \mathcal{W} \subseteq \mathbb{R}^n$ and $f_o(0,0) = 0$. Let us now focus on the open-loop nominal counterpart of the system, namely

$$x(k+1) = f_o\big(x(k), u(k)\big), \quad k \in \mathbb{N}, \quad x(0) = \bar{x}. \tag{1.19}$$

The state and control variables are supposed to satisfy the following constraints: $\forall k \in \mathbb{N}$,

$$x(k) \in X \quad \text{and} \quad u(k) \in U, \tag{1.20}$$

with X and U being compact subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, containing the origin as an interior point.

Definition 9 *For a given control law $u = \phi(x)$, a set $\overline{X} \subseteq X$ is said to be an admissible set for the nominal closed-loop system $x(k+1) = f_o\Big(x(k), \phi\big(x(k)\big)\Big)$ iff $\forall x \in \overline{X}$, $\phi(x) \in U$, $\overline{X}$ is a positively invariant set and $\forall x(0) \in \overline{X}$, $\lim_{k\to+\infty} x(k) = 0$.*

A nominal MPC controller is defined as follows: fix the length of the prediction horizon $N > 0$ and let $u_{0,N-1} = [u(0), u(1), \cdots, u(N-1)] \in U^N$. For a given stage cost $l : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$, a terminal penalty function $V_f : \mathbb{R}^n \to \mathbb{R}_+$ and a terminal set X_f, consider the finite horizon optimal control problem (FHOCP) that consists of minimizing, with respect to $u_{0,N-1}$, the performance index

$$J(\bar{x}, u_{0,N-1}, N) = \textstyle\sum_{k=0}^{N-1} l\big(x(k), u(k)\big) + V_f\big(x(N)\big)$$

subject to

i. the nominal state dynamics (1.19);

ii. the constraints (1.20), $\forall k \in [0, N-1]$;

iii. the terminal state constraint $x(N) \in X_f$.

Let $u^{\text{opt}}_{0,N-1}$ be the optimal control sequence and suppose that only the first element $u^{\text{opt}}_{0,0}(\bar{x})$ of such a sequence is applied, then at the successive time instant the FHOCP is solved again with $\bar{x} = f_o\big(\bar{x}, u^{\text{opt}}_{0,0}(\bar{x})\big)$. This procedure defines a state feedback control law denoted by

$$u = \phi^{\text{MPC}}(x). \tag{1.21}$$

Many results are available on the way to select the parameters that guarantee the stability of the origin for the closed-loop system (1.19), (1.21), see e.g. [13]. Let us recall a standard formulation that turns out to be suitable to our purpose of synthesizing an MPC controller for the nominal system and ensuring closed-loop robustness properties.

Assumption 2 *The function $l(x,u)$ is such that $l(0,0) = 0$ and $l(x,u) \geq \alpha_l(|x|)$, where $\alpha_l(s)$ is a $\mathcal{K}_\infty$-function.*

Assumption 3 *The design parameters V_f and X_f are such that, given an auxiliary control law $u = \phi_f(x)$, one has:*

1. *$X_f \subseteq X$, X_f is closed and $0 \in int(X_f)$;*
2. *$\phi_f(x) \in U$, $\forall x \in X_f$;*
3. *$f_o(x, \phi_f(x)) \in X_f$, $\forall x \in X_f$;*
4. *$0 \leq V_f(x) \leq \beta_{V_f}(|x|)$, $\forall x \in X_f$, with $\beta_{V_f}(s)$ being a $\mathcal{K}_\infty$-function;*
5. *$V_f(f_o(x, \phi_f(x)) - V_f(x) \leq -l(x, \phi_f(x))$, $\forall x \in X_f$.*

Proposition 4 *[12] Let $X^{\text{MPC}}(N)$ be the set of the states such that a solution for the FHOCP exists. Under Assumptions 2 and 3, $X_f \subseteq X^{\text{MPC}}(N)$, the origin is an asymptotically stable equilibrium for the nominal closed-loop system* (1.19), (1.21) *and $X^{\text{MPC}}(N)$ is an admissible set. Moreover, $V(x,N) = J(x, u^{\text{opt}}_{0,N-1}, N)$ is a Lyapunov function for the nominal closed-loop dynamics with $\alpha_1 = \alpha_3 = \alpha_l$, $\alpha_2 = \beta_{V_f}$, $\Gamma = X^{\text{MPC}}(N)$ and $\Omega = X_f$.*

Consider the nominal closed-loop dynamics under the auxiliary control law: by Assumptions 3.4 and 3.5, it follows that $V_f(x)$ is a Lyapunov function with $\alpha_1 = \alpha_3 = \alpha_l$ and $\alpha_2 = \beta_{V_f}$. Hence, the results reported in Section 1.3 allow one to analyze the robustness properties of the closed-loop system in terms of such a $\mathscr{K}_\infty$-functions. By Proposition 4, it also turns out that the $\mathscr{K}_\infty$-functions associated with the Lyapunov function $V(x,N)$ for the nominal closed-loop dynamics under the MPC law are the same as those associated with $V_f(x)$. This means that, within the final set X_f, the MPC controller guarantees the same robustness properties as those which can be proved for the auxiliary law in terms of the functions α_l and β_{V_f}. In particular, the validity of RPI properties for X_f ensures that, under the MPC law, the input and state constraints are robustly satisfied within the final set.
This result, however, does not allow one to conclude on robustness properties out of the final set. Hence, for initial conditions far from the equilibrium, standard but more burdensome methods, such as min-max [10, 17] or constraint tightening [6, 3], should be undertaken.

1.5.1 Auxiliary control law design

According to the previous discussion, a fundamental requirement to guarantee a robust closed-loop dynamics with MPC is that robustness properties are ensured by the adopted auxiliary control law. This fact calls for the need to investigate the way to design auxiliary control laws based on the nominal open-loop model and capable of guaranteeing desired robustness properties for the corresponding closed-loop dynamics. To this aim, stemming from [1], suppose that the mapping f_o describing the nominal dynamics is continuously differentiable in a neighborhood of the equilibrium $(0,0)$. Let

$$\begin{aligned} f_o(x,u) = {} & \left.\frac{\partial f_o}{\partial x}\right|_{x=u=0} x + \left.\frac{\partial f_o}{\partial u}\right|_{x=u=0} u + \pi(x,u) = \\ = {} & A_o x + Bu + \pi(x,u), \end{aligned}$$

where

$$\lim_{|(x,u)| \to 0^+} \frac{|\pi(x,u)|}{|(x,u)|} = 0. \tag{1.22}$$

Assume that the pair (A_o, B) is stabilizable and let the auxiliary law take the form

$$u(k) = Kx(k),$$

with K such that $A_o + BK$ is a Schur matrix. Consider a quadratic stage cost

$$l(x,u) = x'Qx + u'Ru, \quad Q > 0,\ R > 0$$

and let $P > 0$ be the solution of the Lyapunov equation

$$(A_o + BK)'P(A_o + BK) - P = -\beta(Q + K'RK), \tag{1.23}$$

with $\beta > 1$. Hence, consider $V_f(x) = x'Px$ and the vector norm $|\cdot|_P$, then $\alpha_l(s) = cs^2$ and $\beta_{V_f}(s) = s^2$, where $c = \frac{\lambda_{\min}((T^{-1})'QT^{-1})}{\lambda_{\text{Max}}((T^{-1})'PT^{-1})}$ (for some $T \in GL(n, \mathbb{R})$, see equation (1.15)), are so that Assumptions 2 and 3.4 are satisfied. Moreover, for the original nonlinear dynamics, one easily obtains

$$\begin{aligned} V_f\big(f_o(x,Kx)\big) - V_f(x) + l(x,Kx) = \\ = -(\beta-1)x'(Q+K'RK)x + e_\pi(x), \end{aligned}$$

where, in view of (1.22), e_π is such that $\lim_{|x|\to 0^+} \frac{|e_\pi(x)|}{|x|^2} = 0$. Thus, for a sufficiently small $\rho > 0$, Assumption 3 is satisfied with $X_f = \{x \in \mathbb{R}^n | x'Px \le \rho^2\} = \mathscr{B}_\rho$.
Recalling that $\alpha_1(s) = \alpha_3(s) = cs^2$ and $\alpha_2(s) = s^2$, the Ψ-function associated to V_f is then given by

$$\Psi(s) = \left(1 - \sqrt{\tfrac{1-c}{c}}\right)s$$

and it is a $\mathscr{K}_\infty$-function if and only if $c > \frac{1}{2}$, that is

$$\frac{\lambda_{\min}((T^{-1})'QT^{-1})}{\lambda_{\mathrm{Max}}((T^{-1})'PT^{-1})} > \frac{1}{2}. \tag{1.24}$$

If condition (1.24) is satisfied, then Theorem 3 ensures that, with both the auxiliary law and MPC designed on the nominal system (1.19), the closed loop system (1.18), (1.21) is $(\mathscr{B}_r, \mathscr{B}_{\Psi(r)})$-ISS, $\forall r \le \rho$.
This result raises the issue of the existence and selection of the design parameters K, Q, R and β so that condition (1.24) is satisfied. To this end, the following result holds:

Proposition 5 *If $T \in GL(n,\mathbb{R})$ and $K \in \mathbb{R}^{m\times n}$ are so that $|A_o+BK|^2_{T'T} < \frac{1}{2}$, then A_o+BK is Schur and there exists a sufficiently small $\varepsilon > 0$ such that, with $Q = T'T$, $R = \varepsilon I$ and $\beta = 1+\varepsilon$, condition* (1.24) *is satisfied.*

Proof. The matrix A_o+BK is Schur because $\rho(A_o+BK) \le |A_o+BK|_{T'T} < \frac{\sqrt{2}}{2} < 1$.
Let $Q_1 = \beta(Q+K'RK)$ and $A_c = A_o+BK$. In the new coordinates $\hat{x} = Tx$, one has $\hat{A}_c = TA_cT^{-1}$, $\hat{Q} = (T^{-1})'QT^{-1}$, $\hat{Q}_1 = (T^{-1})'Q_1T^{-1}$, $\hat{P} = (T^{-1})'PT^{-1}$. Take Q, R and β as in the assumptions, then $\hat{Q} = I$ and, being $|\hat{A}_c|^2_2 = |A_c|^2_{T'T} < \frac{1}{2}$, there exists a sufficiently small $\varepsilon > 0$ such that

$$1 - \frac{\lambda_{\mathrm{Max}}(\hat{Q}_1)}{2\lambda_{\min}(\hat{Q})} > |\hat{A}_c|^2_2. \tag{1.25}$$

Equation (1.23) rewrites as $\hat{P} = \hat{Q}_1 + \hat{A}_c'\hat{P}\hat{A}_c$. Hence, $\lambda_{\mathrm{Max}}(\hat{P}) = |\hat{P}|_2 \le |\hat{Q}_1|_2 + |\hat{A}_c'\hat{P}\hat{A}_c|_2 \le \lambda_{\mathrm{Max}}(\hat{Q}_1) + \lambda_{\mathrm{Max}}(\hat{P})|\hat{A}_c|^2_2$ so that

$$\lambda_{\mathrm{Max}}(\hat{P}) \le \frac{\lambda_{\mathrm{Max}}(\hat{Q}_1)}{1-|\hat{A}_c|^2_2}. \tag{1.26}$$

because $|\hat{A}_c|^2_2 < 1$. Thus,

$$\frac{\lambda_{\min}(\hat{Q})}{\lambda_{\mathrm{Max}}(\hat{P})} \overset{(a)}{\ge} \frac{\lambda_{\min}(\hat{Q})}{\lambda_{\mathrm{Max}}(\hat{Q}_1)}(1-|\hat{A}_c|^2_2) \overset{(b)}{>} \frac{1}{2},$$

where inequalities (a) and (b) follow by inequalities (1.26) and (1.25), respectively. ■
According to Proposition 5, for a given $T \in GL(n,\mathbb{R})$, the problem of the selection of the design parameters so that condition (1.24) holds is then reduced to determine K such that $|T(A_o+BK)T^{-1}|^2_2 < \frac{1}{2}$. Such a control design issue can be expressed in the form of the following LMI problem: find $K \in \mathbb{R}^{m\times n}$ such that

$$\begin{bmatrix} \frac{\sqrt{2}}{2}I & T(A_o+BK)T^{-1} \\ (T^{-1})'(A_o'+K'B')T' & \frac{\sqrt{2}}{2}I \end{bmatrix} > 0.$$

If there exists K such that $\rho(A_o+BK) < \frac{\sqrt{2}}{2}$, a suitable choice for T makes such an LMI feasible. In fact, it is well known that, $\forall \varepsilon > 0$, $\exists T \in GL(n,\mathbb{R})$ such that $|A_o+BK|_{T'T} < \rho(A_o+BK) + \varepsilon$.

1.6 Conclusions

In this paper, it has been shown that, under mild assumptions, the design of an MPC law for a nominal model guarantees also robustness in perturbed conditions. This has been proved by first deriving a number of results for systems characterized by a not necessarily continuous dynamic equation and subject to additive disturbances. It is believed that these results can be of wider applicability, including the study of other control synthesis techniques than MPC. More general results, accounting also for state dependent disturbances, are currently under development.

Chapter 2

Fully decentralized nominal MPC

The contents of this chapter have been developed by Daniel Limón (Dpto. Ingenieria de Sistemas y Automatica Escuela Superior de Ingenieros, Universidad de Sevilla).

2.1 Introduction

Decentralized model predictive control (MPC) techniques are of great interest in the process industry due to topological reasons and the possible limited exchange of data between subsystems. Moreover, this decentralized control technique often has the advantage to reduce the possibly large-size optimization problem to the solution of smaller ones potentially more tractable.
Distributed MPC algorithms can be developed (i) assuming that there exists exchange of information between the subsystems, or (ii) considering that there not exists any information exchange yielding to a fully decentralized control structure. In this work the second case is considered, that is, a fully decentralized MPC. In this case, the possible interactions between subsystems are considered as unknown disturbances that the controller must accomplish. The design of a fully decentralized MPC can be done relying on a robust design of each predictive controllers [29].
Model predictive control is one of the few techniques capable to control a nonlinear plant guaranteeing asymptotic stability to the target operating point fulfilling hard constraints on the state and input. The control law is implicitly derived from the solution of an optimization problem at each sampling time and the receding horizon technique ([13]). In the case that the prediction model differs from the real plant, then the effect of the uncertainty must be considered. Under some mild assumptions, the predictive control law ensure robust stability in the case that the uncertainty is small enough ([22, 25]). In other case, the uncertainty model must be considered in the controller calculation in order to provide robust stability and robust constraint satisfaction. In this case particularly interesting are those approaches that provide robustness based on the solution of a *nominal* optimization problem. Input-to-state stability appears as a suitable framework for the robust stability analysis while constraint satisfaction can be ensured by means of approximations of the reachable sets. See [26] and the references there in for a survey on this topic.
In [29] a decentralized min-max MPC is proposed. Stability of the whole plant is achieved relying on the ISS property of each single min-max MPC controller and assuming certain bounds on the coupling terms. In this work we extend this result to the case of nominal MPC, which avoids the computational complexity of the solution of the min-max optimization problem.
In this work we present the methodology to design the nominal MPC for each subsystem. Under a certain design, which generalizes [24], the nominal MPC can ensure ISS of the system with a less

conservative stability margin. The uncertainty is modeled as a parametric uncertain signal, not as an additive disturbance. Assuming that the model function is uniformly continuous, enhanced design of the robust controller is achieved: in the calculation of the constraints of the optimization problem and in the stabilizing conditions. The obtained stabilizing design of the controller results particularly interesting to relax the terminal conditions for a certain class of model functions yielding to a less conservative control law.

Notation and basic definitions:

Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$ and $\mathbb{Z}_{\geq 0}$ denote the real, the non-negative real, the integer and the non-negative integer numbers, respectively. Given two integers $a,b \in \mathbb{Z}_{\geq 0}$, $\mathbb{Z}_{[a,b]} \triangleq \{j \in \mathbb{Z}_{\geq 0} : a \leq j \leq b\}$. Given two vectors $x_1 \in \mathbb{R}^a$ and $x_2 \in \mathbb{R}^b$, $(x_1,x_2) \triangleq [x_1', x_2']' \in \mathbb{R}^{a+b}$. A norm of a vector $x \in \mathbb{R}^a$ is denoted as $|x|$. Given a signal $w \in \mathbb{R}^a$, the signal's sequence is denoted by $\mathbf{w} \triangleq \{w(0), w(1), \cdots\}$ where the cardinality of the sequence is inferred from the context. $\mathbf{0}$ denotes a suitable signal's sequence taking a null value. If a sequence depends on a parameter, as $\mathbf{w}(x)$, $w(j,x)$ denotes its j-th element. The sequence $\mathbf{w}_{[\tau]}$ denotes the truncation of sequence $\mathbf{w}$, i.e. $w_{[\tau]}(j) = w(j)$ if $j \leq \tau$ and $w_{[\tau]}(j) = 0$ if $j > \tau$. For a given sequence, we denote $\|w\| \triangleq \sup_{k \geq 0}\{|w(k)|\}$. The set of sequences $\mathbf{w}$, whose elements $w(j)$ belong to a set $W \subseteq \mathbb{R}^a$ is denoted by $\mathscr{M}_W$. For a compact set A, $A^{sup} \triangleq \sup_{a \in A}\{|a|\}$.

Consider a function $f(x,y) : \mathbb{R}^a \times \mathbb{R}^b \to \mathbb{R}^c$, f is said to be uniformly continuous in x for all $x \in A$ and $y \in B$ if for all $\varepsilon > 0$, a $\delta(\varepsilon) > 0$ exists such that $|f(x_1,y) - f(x_2,y)| \leq \varepsilon$ for all $x_1, x_2 \in A$ with $|x_1 - x_2| \leq \delta(\varepsilon)$ and for all $y \in B$. For a given set $A \subset \mathbb{R}^a$, the range of the function is $f(A,y) \triangleq \{f(x,y) : x \in A\} \subset \mathbb{R}^c$.

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathscr{K}$ (or a "$\mathscr{K}$-function") if it is continuous, strictly increasing and $\gamma(0) = 0$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathscr{K}_\infty$ if it is a $\mathscr{K}$-function and $\gamma(s) \to +\infty$ as $s \to +\infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathscr{KL}$ if, for each fixed $t \geq 0$, $\beta(\cdot,k)$ is of class $\mathscr{K}$, for each fixed $s \geq 0$, $\beta(s,\cdot)$ is decreasing and $\beta(s,k) \to 0$ as $k \to +\infty$. Consider a couple of $\mathscr{K}$-functions σ_1 and σ_2, then $\sigma_1 \circ \sigma_2(s) \triangleq \sigma_1(\sigma_2(s))$, besides $\sigma_1^j(s)$ denotes the j-th composition of σ_1, i.e. $\sigma_1^{j+1}(s) = \sigma_1 \circ \sigma_1^j(s)$ with $\sigma_1^1(s) \triangleq \sigma_1(s)$. A function $V : \mathbb{R}^a \to \mathbb{R}_{\geq 0}$ is called positive definite if $V(0) = 0$ and there exists a $\mathscr{K}$-function α such that $V(x) \geq \alpha(|x|)$.

2.2 Problem statement

In this work it is considered that the subsystem to be controlled is described by a discrete-time invariant nonlinear difference equation as follows

$$x(k+1) \quad = \quad f(x(k), u(k), w(k)), \; k \geq 0 \tag{2.1}$$

where $x(k) \in \mathbb{R}^n$ is the system state, $u(k) \in \mathbb{R}^m$ is the current controlled variable and $w(k) \in \mathbb{R}^p$ is a signal which models mismatches between the real plant and the model, that is, the unknown coupling terms. The origin is an equilibrium point for the plant (i.e. $f(0,0,0) = 0$) which is the control target. The solution of system (2.1) at sampling time k for the initial state $x(0)$, a sequence of control inputs $\mathbf{u}$ and uncertainty signal $\mathbf{w}$ is denoted as $\phi(k, x(0), \mathbf{u}, \mathbf{w})$, where $\phi(0, x(0), \mathbf{u}, \mathbf{w}) = x(0)$. It is assumed that there is no trajectory $\phi(k, x(0), \mathbf{u}, \mathbf{w})$ that exhibits finite escape time for any $x(0)$, $\mathbf{u}$ and $\mathbf{w}$. It is also assumed that the state of the plant $x(k)$ can be measured at each sample time.

It is considered that the uncertainty signal $w(k)$ lies in a known ball $\mathscr{W} = \{w : |w| \leq \mu\}$. Furthermore, the control input and state of the plant must fulfill the following hard constraint:

$$(x(k), u(k)) \in \mathscr{Z} \tag{2.2}$$

where $\mathscr{Z} \subseteq \mathbb{R}^{n+m}$ is closed and contains the origin in its interior.
The model function is assumed to be uniformly continuous in all its arguments in the set $\mathscr{Z} \times \mathscr{W}$. Then, there are three $\mathscr{K}$-functions σ_x, σ_u and σ_w such that

$$\begin{aligned} |f(x_1,u_1,w_1) - f(x_2,u_2,w_2)| \quad \leq \quad & \sigma_x(|x_1 - x_2|) \\ & +\sigma_u(|u_1 - u_2|) \\ & +\sigma_w(|w_1 - w_2|) \end{aligned} \tag{2.3}$$

for all (x_1,u_1,w_1) and (x_2,u_2,w_2) in $\mathscr{Z} \times \mathscr{W}$.
The nominal model of the plant (2.1) denotes the system considering zero-disturbance and it is given by

$$\tilde{x}(k+1) \quad = \quad \tilde{f}(\tilde{x}(k),u(k)),\ k \geq 0 \tag{2.4}$$

where $\tilde{f}(x,u) \triangleq f(x,u,0)$. The solution to this equation for a given initial state $x(0)$ is denoted as $\tilde{\phi}(k,x(0),\mathbf{u}) \triangleq \phi(k,x(0),\mathbf{u},\mathbf{0})$.
The aim of the work is to design a model predictive controller based on nominal predictions such that the controlled plant is robustly stable while satisfying the constraints throughout the evolution. In the following sections, the stability notion used in this work is briefly introduced: the regional input-to-state stability.

2.2.1 Regional input-to-state stability (ISS)

The existence of constraints limits the domain where the system can be stabilized. Then, a regional definition of the stability notions must be considered. In this work, robust stability is studied resorting in the notion of input-to-state stability ([32, 23]). ISS has demonstrated to be a useful framework to analyze robust stability of predictive controllers (see [26] and the references there in).
Consider that the system (2.1) is controlled by the law $u = \kappa(x)$ leading the following closed-loop system

$$x^+ \quad = \quad f_\kappa(x,w) \triangleq f(x,\kappa(x),w) \tag{2.5}$$

$$x \quad \in \quad X_\kappa \triangleq \{x \in \mathbb{R}^n : (x,\kappa(x)) \in \mathscr{Z}\} \tag{2.6}$$

Now, some definitions and well-known results on regional ISS are summarized.

Definition 10 (Robust positively invariant (RPI) set) *A set $\Gamma \subseteq \mathbb{R}^n$ is a robust positively invariant (RPI) set for system (2.5) if $f_\kappa(x,w) \in \Gamma$, for all $x \in \Gamma$ and all $w \in \mathscr{W}$. Furthermore, if $\Gamma \subseteq X_\kappa$, then Γ is called admissible RPI set.* □

Notice that the fact that the RPI set Γ is admissible ensures the robust satisfaction of the constraints since for any initial $x_0 \in \Gamma$, $\Phi(k,x_0,\mathbf{w}) \in \Gamma \subseteq X_\kappa$ for all $k \in \mathbb{Z}_{\geq 0}$ and $\mathbf{w} \in \mathscr{M}_\mathscr{W}$.

Definition 11 (Regional *ISS* in Γ) *Let $\Gamma \subseteq \mathbb{R}^n$ be an admissible RPI for system* (2.5) *including the origin as an interior point. The system* (2.5) *is input-to-state stable (ISS) in Γ if there exist a $\mathscr{KL}$-function β and a $\mathscr{K}$-function σ such that*

$$|\phi_\kappa(j,x(0),\mathbf{w})| \quad \leq \quad \beta(|x(0)|,j) + \sigma(||\mathbf{w}_{[j-1]}||) \tag{2.7}$$

for all $x(0) \in \Gamma$, $\mathbf{w} \in \mathscr{M}_\mathscr{W}$ and $j \in \mathbb{Z}_{\geq 0}$.

ISS can be determined by means of a Lyapunov-like condition ([23, 27]), as follows.

Definition 12 *(ISS-Lyapunov function in Γ) Let Γ be a RPI set containing the origin in its interior. A function $V\colon \mathbb{R}^n \to \mathbb{R}_{\geq}0$ is called an ISS-Lyapunov function in Γ for system (2.5) if there exist a compact set $\Omega \subseteq \Gamma$ (including the origin as an interior point), suitable $\mathscr{K}_\infty$-functions α_1, α_2, α_3 and $\mathscr{K}$-function λ such that:*

$$V(x) \geq \alpha_1(|x|), \quad \forall x \in \Gamma \tag{2.8}$$

$$V(x) \leq \alpha_2(|x|), \quad \forall x \in \Omega \tag{2.9}$$

and for all $x \in \Gamma$ and $w \in \mathscr{W}$, the following condition holds

$$V(f_\kappa(x,w)) - V(x) \leq -\alpha_3(|x|) + \lambda(|w|) \tag{2.10}$$

□

Based on this Lyapunov-like functions, the following stability theorem can be derived ([23, 27]):

Theorem 1 *If system (2.5) admits an ISS-Lyapunov function in Γ then it is ISS in Γ.*

2.3 Proposed robust MPC

2.3.1 Semi-feedback approach

The most simple robust MPC formulations derive the control law from the solution of an optimization problem based on open-loop predictions of the uncertain system evolution. This open-loop scheme results to be very conservative from both a performance and domain of attraction points of view (see [13, Section 4]). In order to reduce this conservativeness, a closed-loop (or feedback) formulation of the MPC has been proposed ([31]). In this case, control policies instead of control actions are taken as decision variables, yielding to an infinite dimensional optimization problem that is in general very difficult to solve and for which there exists few efficient algorithm in the literature in the case of linear systems ([28, 21]). A practical formulation between these two approaches is the so-called semi-feedback formulation, where a family of parameterized control laws is used ([19, 20]). Thus the decision variables are the sequence of the parameters of the control laws, and hence the optimization problem is a finite-dimensional mathematical programming problem.
Consider that the control actions are derived from a given family of controllers parameterized by $v \in \mathbb{R}^s$,

$$u(k) = \pi(x(k), v(k))$$

which is assumed to be uniformly continuous in its domain. The family of control laws is typically chosen as an affine function of the state ([19]). Thus, system (2.1) is transformed in

$$x(k+1) = f_\pi(x(k), v(k), w(k)),\ k \geq 0 \tag{2.11}$$

where $f_\pi(x,v,w) \triangleq f(x, \pi(x,v), w)$. Notice that v plays the role of the input of the modified system. The solution of this equation is denoted as $\phi_\pi(k,x,\mathbf{v},\mathbf{w})$. The nominal model of system (2.11) is denoted as $\tilde{f}_\pi(x,v) \triangleq f_\pi(x,v,0)$ and its solution as $\tilde{\phi}_\pi(k,x,\mathbf{v}) \triangleq \phi_\pi(k,x,\mathbf{v},\mathbf{0})$. Analogously, the constraints can be rewritten as

$$(x(k), v(k)) \in \mathscr{Z}_\pi \tag{2.12}$$

where $\mathscr{Z}_\pi$ is such that $(x, \pi(x,v)) \in \mathscr{Z}$ for all $(x,v) \in \mathscr{Z}_\pi$.

2.3.2 Nominal model predictive control

The proposed predictive controller is based on the nominal prediction of the trajectories and follows the standard formulation of the MPC ([13]). The control law is derived from the solution of the following mathematical programming problem $P_N(x)$ parameterized in the current state x.

$$\min_{\mathbf{v}} \quad \sum_{j=0}^{N-1} L_\pi(\tilde{x}(j),v(j)) + V_f(\tilde{x}(N)) \tag{2.13}$$

$$s.t. \quad \tilde{x}(j) = \tilde{\phi}_\pi(j,x,\mathbf{v}),\ j \in \mathbb{Z}_{[0,N]} \tag{2.14}$$

$$(\tilde{x}(j),v(j)) \in \mathscr{Z}_\pi(j),\ j \in \mathbb{Z}_{[0,N-1]} \tag{2.15}$$

$$\tilde{x}(N) \in \mathscr{X}_f \tag{2.16}$$

where $L_\pi(x,v) \triangleq L(x,\pi(x,v))$ and $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ is the stage cost function, $V_f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is the terminal cost function. The sequence of constraint sets $\{\mathscr{Z}_\pi(j)\}$ will be defined later on and $X_f \subseteq \mathbb{R}^n$ is the terminal region. It is assumed that $P_N(x)$ is feasible in a non-empty region denoted $\mathscr{X}_N$. For each $x \in X_N$, the argument of $P_N(x)$ is denoted $\mathbf{v}^*(x)$ and the optimal cost is $V_N^*(x)$. The MPC control law derives from the application of the solution in a receding horizon manner $\kappa_N(x) = v^*(0;x)$ and it is defined for all $x \in \mathscr{X}_N$.

2.3.3 Robust design of the proposed controller

The proposed controller is based on the availability of two sequence of sets $\{\mathscr{R}(j)\}$ and $\{\mathscr{F}(j)\}$ that are assumed to be calculated off-line (see next section). The sequence $\{\mathscr{F}(j)\}$ is related with the free response of the nominal system and must satisfy the following hypothesis:

Assumption 4 *The sequence of sets $\{\mathscr{F}(j)\}$ is such that: For every $(x,\mathbf{v})$, $\tilde{\phi}_\pi(k,\hat{x},\mathbf{v}) \in \tilde{\phi}_\pi(k,x,\mathbf{v}) \oplus \mathscr{F}(k)$ for all $\hat{x}$ such that $|\hat{x}-x| \leq \sigma_w(\mu)$.*

On the other hand, the sequence $\{\mathscr{R}_j\}$ is related to the reachable sets, that is, the sequence of possible trajectories due to the effect of the disturbances. This sequence must satisfy the following conditions

Assumption 5 *The sequence of sets $\{\mathscr{R}(j)\}$ is such that:*

1. *For every $(x,\mathbf{v})$, $\phi_\pi(k,x,\mathbf{v},\mathbf{w}) \in \tilde{\phi}_\pi(k,x,\mathbf{v}) \oplus \mathscr{R}(k)$ for all $\mathbf{w} \in \mathscr{M}_W$*

2. *$\mathscr{F}(j) \oplus \mathscr{R}(j) \subseteq \mathscr{R}(j+1)$*

The first condition states that each set of the sequence is an outer bound of the effect of the uncertainty throughout the trajectory, while the second condition ensures that the sequence is monotone. This fact will be more clearly demonstrated in the proof of lemma 1. Practical methods to calculate the proposed sequences are presented in the following section.
Since sequence of sets $\{\mathscr{R}(j)\}$ provides an estimation of the effect of the disturbance with respect to the nominal predictions, this can be used to counteract the effect of the disturbances in the constraint satisfaction. This is done by using a sequence of tighter constraint sets $\{\mathscr{Z}_\pi(j)\}$ defined as follows:

Definition 13 *Let the sequence $\{\mathscr{Z}_\pi(j)\}$ be defined as follows*

$$\mathscr{Z}_\pi(j) = \mathscr{Z} \ominus (\mathscr{R}(j) \times \{0\})$$

On the other hand, the terminal constraint set $\mathscr{X}_f$ must satisfy the following assumption

Assumption 6 *The input $v_f \in \mathbb{R}^s$ and the sets Ω and $\mathscr{X}_f$ are such that*

(i) Ω *and* $\mathscr{X}_f$ *are invariant sets for the system* $\tilde{x}^+ = \tilde{f}_\pi(\tilde{x}, v_f)$

(ii) $\Omega \times \{v_f\} \subseteq \mathscr{Z}_\pi(N-1)$

(iii) $\mathscr{X}_f \oplus \mathscr{F}(N-1) \subseteq \Omega$

(iv) $\tilde{f}_\pi(\tilde{x}, v_f) \in \mathscr{X}_f$, *for all* $\tilde{x} \in \Omega$.

Notice that this assumption requires that Ω is an invariant set for the nominal system and $\mathscr{X}_f$ a set where the system evolves in one step. The main restriction is that Ω must be a contractive invariant set such that $\tilde{f}_\pi(\Omega, v_f) \oplus \mathscr{F}(N-1) \subseteq \Omega$. This implicitly states that Ω is a robust positively invariant set for the system $x^+ = f_\pi(x, v_f) + \omega$ where $\omega \in \mathscr{F}(N-1)$. In the case that the control law π makes the system asymptotically stable in $(x,v) \in \mathscr{Z}_\pi$, which is usual for simple systems such that linear systems ([19]) or feedback linearizable system ([30]), set $\mathscr{F}(N-1)$ can be arbitrarily small for large enough prediction horizon. This relaxes the standard assumption on the terminal constraint, that must be a robust positively invariant set for the whole uncertainty set.
This proposed method to design the constraints of the optimization problem P_N has been chosen in order to ensure the robust feasibility of the controller, as it is demonstrated in the following lemma.

Lemma. 1 *Consider the system* (2.11) *and the sequence of sets* $\{\mathscr{Z}_\pi(j)\}$ *based on the sequences of sets* $\{\mathscr{R}(j)\}$ *and* $\{\mathscr{F}(j)\}$ *which satisfy assumptions 4 and 5. Let the triplet* $(v_f, \Omega, \mathscr{X}_f)$ *fulfill assumption 6. Consider now a feasible state* $x \in \mathscr{X}_N$ *and* $\mathbf{v}^*$ *the argument of* $P_N(x)$*. Let* x^+ *be the uncertain successor state and define the sequence of inputs* $\mathbf{v}^+ \triangleq \{v^*(1), \cdots, v^*(N-1), v_f\}$*. Then the following properties hold.*

1. $(\tilde{\phi}_\pi(j, x^+, \mathbf{v}^+), v^+(j)) \in \mathscr{Z}_\pi(j)$
2. $\tilde{\phi}_\pi(N, x^+, \mathbf{v}^+) \in \mathscr{X}_f$

2.3.4 Calculation of the sequence of sets

The sequence of sets $\{\mathscr{F}(j)\}$ and $\{\mathscr{R}(j)\}$ provides outer bounds on the effect of the uncertainty throughout the prediction, then these can be calculated by methods that provides guaranteed prediction of the uncertain system ([18, 30, 26]). Among these, it is worth to cite those based on polytopic algorithms, interval arithmetics, zonotopic methods or DC-programming based techniques.
In this work we provide a simpler, although probably more conservative method, based on the uniform continuity of the model function.

Lemma. 2 *Let a system be given by model* (2.1) *and let define the following sets:*

$$\mathscr{F}(j) \triangleq \{x \in \mathbb{R}^n : |x| \leq \sigma_x^j \circ \sigma_w(\mu)\} \tag{2.17}$$

$$\mathscr{R}(j) \triangleq \{x \in \mathbb{R}^n : |x| \leq c_j(\mu)\} \tag{2.18}$$

where $c_j(\mu)$ is given by the following recursion

$$c_j(\mu) = max\{ \begin{array}{l} \sigma_w(\mu) + \sigma_x \circ c_{j-1}(\mu), \\ c_{j-1}(\mu) + \sigma_x^{j-1} \circ \sigma_w(\mu)\} \end{array} \tag{2.19}$$

with $c_1(\mu) = \sigma_x(\mu)$.
Then the sequence of sets $\{\mathscr{F}(j)\}$ and $\{\mathscr{R}(j)\}$ satisfy the assumptions 4 and 5.

As it can be seen, these sets can be easily calculated off-line once provided the bounding functions. In the case that Lipschitz continuity is exploited to derive the bounding functions, the resulting sets are equal to those presented in [24]. Notice that if the uniform continuity is exploited, tighter (non-linear) bounding functions can be used, and hence less conservative results will be obtained.

2.4 Input-to-state stability of the controlled system

In the previous sections, conditions on the constraints of the optimization problem $P_N(x)$ that suffices to ensure robust feasibility are provided. However these conditions are not sufficient to derive robust stability of the closed-loop system. To this aim, the following additional assumptions are required.

Assumption 7

1. *Let the stage cost function $L_\pi(x,v)$ be a definite positive function in (x,v) uniformly continuous in $\mathscr{Z}_\pi$ such that*

$$\begin{aligned} L_\pi(x,v) &\geq \alpha_L(|x|) \\ |L_\pi(x_1,v_1)-L_\pi(x_2,v_2)| &\leq \lambda_x(|x_1-x_2|)+\lambda_v(|v_1-v_2|) \end{aligned}$$

   where α_L, λ_x and λ_v are $\mathscr{K}$-functions.

2. *Let the terminal cost function $V_f(x)$ be a definite positive function uniformly continuous in Ω (see assumption 6) such that*

$$\begin{aligned} \alpha_V(|x|) \leq V_f(x) &\leq \beta_V(|x|) \\ V_f(\tilde{f}_\pi(x,v_f))-V_f(x) &\leq -L_\pi(x,v_f) \\ |V_f(x_1)-V_f(x_2)| &\leq \delta(|x_1-x_2|) \end{aligned}$$

These assumptions are standard for the stabilizing design of nominal MPC ([13]). The only additional requirement is the uniform continuity of the functions. Based on this, stability is stated in the following theorem.

Theorem 2 *Consider that assumptions 4 ,5, 6 and 7, hold. Then the system* (2.1) *controlled by $\kappa_{MPC}(x)=\pi(x,\kappa_N(x))$ is ISS in $\mathscr{X}_N$ and satisfies the constraints throughout the evolution.*

2.5 Conclusions

This work has demonstrated that outer estimation of the reachable sets can be used to derive robust stabilizing predictive controller based on nominal predictions. This class of controllers are appealing from a practical point of view since can be constructed from standard nominal MPC. On the other hand, the open-loop nature of the problem may yield to the results to be useful only for small uncertainties. In order to reduce this effect, semi-feedback approach is proposed. This is a simple and practical method, but requires an analysis of the system to be controlled in order to find a nice family of control laws.

Based on the uniform continuity of the model function and the defining functions of the MPC, sufficient conditions for input-to-state stability has been proposed. Moreover, uniform continuity can also be exploited to calculate the sequence of sets necessary for the design of the proposed controller.

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